

The Higgs Mechanism in Heterotic Orbifolds

Stefan Förste^a, Hans Peter Nilles^b and Akin Wingerter^c

^a *Institute for Particle Physics Phenomenology (IPPP)
South Road, Durham DH1 3LE, United Kingdom*

^b *Physikalisches Institut, Universität Bonn
Nussallee 12, D-53115 Bonn, Germany*

^c *Department of Physics, The Ohio State University
191 W. Woodruff Ave., Columbus, OH 43210, USA*

Abstract

We study spontaneous gauge symmetry breaking in the framework of orbifold compactifications of heterotic string theory. In particular we investigate the electroweak symmetry breakdown via the Higgs mechanism. Such a breakdown can be achieved by continuous Wilson lines. Exploiting the geometrical properties of this scheme we develop a new technique which simplifies the analysis used in previous discussions.

1 Introduction

An outstanding problem of string theory is the construction of realistic models. One of the first attempts to solve that problem is provided by orbifold compactifications of the heterotic string [1, 2]. Such constructions allow us to obtain four dimensional effective theories with $N = 1$ supersymmetry. The massless spectrum can be determined explicitly and naturally comes out to be chiral. Further progress has been made by the introduction of discrete Wilson lines [3]. That mechanism is a suitable tool in order to control the number of families as well as to reduce the unbroken gauge group down to groups like $SU(3) \times SU(2) \times U(1)$ [4]. Subsequently many models have been studied [5–25]. For a review see e.g. [26]. (Related free fermionic constructions [27–31] will not be considered in the present paper.)

Bottom up approaches to physics beyond the Standard Model started to discuss the existence of extra dimensions seriously during the past years. One of the motivations is that models with extra dimensions allow to keep attractive features of grand unification while some severe problems such as the doublet triplet splitting problem can be eliminated. Models of that type are known as orbifold GUTs [32–37]. See [38] for a review containing more references.

The observations within bottom up approaches triggered renewed interest in heterotic orbifold constructions. Indeed, combining the top down string theory constructions with the bottom up orbifold GUT models has the prospect of providing very useful results. Field theory constructions can be put on firmer grounds since string theory yields consistent prescriptions how to deal with orbifold geometries and serves as a UV completion. Guidelines in GUT orbifold model building can emerge from string theory. On the other hand, results of bottom up approaches improve the situation for attempts to connect string theory to the real world. From all possible candidates heterotic orbifolds are the most natural ones incorporating an effective orbifold GUT picture. Moreover, such a picture is obtained in models where the orbifold possesses fixed tori. Together with the requirement of unbroken $N = 1$ supersymmetry this restricts the orbifold group to be either $\mathbb{Z}_M \times \mathbb{Z}_N$ or \mathbb{Z}_K with K not prime. Recent investigations focus on $\mathbb{Z}_2 \times \mathbb{Z}_2$ [39] and \mathbb{Z}_6 -II [40–43] models.

The present paper is devoted to symmetry breaking by continuous Wilson lines within that class of models. In [44] and more recently in [45] it was demonstrated that continuous Wilson lines are most easily included if the orbifold group is embedded into the gauge group as a rotation of the root lattice. In this case the Wilson line can be taken to point along Cartan directions. It was, however, also encountered that it can be complicated to identify a consistent algebra automorphism which is induced by the rotational embedding of the orbifold group.

In section 2 we explore to what extend the rotational embedding of the orbifold group into the gauge group is essential. We will find that in the field theory picture it is possible to describe continuous Wilson lines also when the orbifold group is embedded with the usual shift leaving all Cartan generators invariant and multiplying root generators with a phase. Thus the shift embedding refers to a particular choice of the Cartan-Weyl basis.

We will confirm the geometric picture described in [45] from a slightly different angle.

At all fixed points the orbifold is defined by a shift embedding. However, the choice of the Cartan-Weyl basis can differ from fixed point to fixed point. In particular the embedding of the Cartan torus into the gauge group can be different. In such a case the rank of the unbroken gauge group will be reduced.

In section 3 we will rederive the particular symmetry breakings discussed in [45] using the new technique. This will allow us to present the projection patterns for bulk matter in the considered model.

The rest of the paper is devoted to electroweak symmetry breaking via continuous Wilson lines. After discussing our general strategy in section 4 we will consider a \mathbb{Z}_6 -II model in section 5 and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ model in section 6. A summary and implications of our findings for string theoretical model building will be presented in a concluding section 7. A couple of appendices provides technical details and supplementary informations.

2 Wilson Lines in Shift Embeddings

For simplicity we will discuss a \mathbb{Z}_2 orbifold GUT with one extra dimension in this section. The \mathbb{Z}_3 example with two extra dimensions is presented in appendix A. As far as the computation of the massless spectrum is concerned a lift to string theory should be also straightforward. As will become clear shortly one can employ the method of constructing fixed point equivalent models [48] which are all shift embedded without Wilson lines.

The five dimensional theory is a gauge theory with gauge group G . The coordinate of the extra dimension is called x^5 . An effective four dimensional theory arises upon compactification on an orbicircle S^1/\mathbb{Z}_2 . This geometry is obtained by identifying points on the real line which are mapped onto each other under an element of the space group. The space group is generated by a \mathbb{Z}_2

$$(\theta, 0) : x^5 \rightarrow -x^5 \quad (1)$$

and the S^1 compactification

$$(1, e) : x^5 \rightarrow x^5 + 2\pi R. \quad (2)$$

Fixed points under the \mathbb{Z}_2 action are points which differ from their \mathbb{Z}_2 image by an integer multiple of $2\pi R$, i.e. they are identical on S^1 . On S^1 there are two fixed points given by

$$x^5 = 0 \text{ and } x^5 = \pi R. \quad (3)$$

We specify the gauge group G by choosing a Cartan-Weyl basis, i.e. a set of Cartan generators

$$H_i, \quad i = 1, \dots, r \quad (4)$$

and a set of root generators

$$E_{\alpha_k}, \quad k = 1, \dots, \dim(G) - r, \quad (5)$$

where r denotes the rank of the gauge group.

The shift embedding of the \mathbb{Z}_2 is defined as follows

$$(\theta, 0) : \begin{array}{ll} H_i & \rightarrow H_i, \\ E_{\alpha_k} & \rightarrow \exp\{2\pi i \alpha_k \cdot V\} E_{\alpha_k}, \end{array} \quad \begin{array}{l} i = 1, \dots, r \\ k = 1, \dots, \dim(G) - r \end{array} \quad (6)$$

where V is an r dimensional Vector with components V^i . The action on the root generators can be alternatively described as

$$E_\alpha \rightarrow e^{2\pi i V^i H_i} E_\alpha e^{-2\pi i V^i H_i}. \quad (7)$$

The requirement that $(\theta^2, 0) = (1, 0)$ leaves G invariant yields the condition that $2V$ should be an element of the coroot lattice of G .

The effect of a Wilson line is that the embedding¹ of (θ, e) contains an additional constant gauge transformation $t \in G$

$$(\theta, e) : \begin{array}{ll} H_i & \rightarrow t H_i t^{-1}, \\ E_{\alpha_k} & \rightarrow \exp\{2\pi i \alpha_k \cdot V\} t E_{\alpha_k} t^{-1}, \end{array} \quad \begin{array}{l} i = 1, \dots, r \\ k = 1, \dots, \dim(G) - r \end{array} \quad (8)$$

Applying this transformation twice yields for example

$$(\theta, e)^2 : H_i \rightarrow \tilde{t} t H_i t^{-1} \tilde{t}^{-1}, \quad (9)$$

where \tilde{t} is the \mathbb{Z}_2 image of t

$$\tilde{t} = e^{2\pi i V^i H_i} t e^{-2\pi i V^i H_i}. \quad (10)$$

Consistency requires²

$$\tilde{t} t = \pm 1. \quad (11)$$

The lower sign appears because gauge fields transform in the adjoint representation of G . Since later we may want to add matter transforming in different representations we will ignore this possibility in the following. First, let us discuss the known case where t is in the Cartan subgroup. Then, (11) leads to a quantisation condition

$$t \in \text{CSG} \implies t^2 = 1. \quad (12)$$

Writing t as

$$t = e^{2\pi i T}, \quad T = a^i H_i. \quad (13)$$

the condition $t^2 = 1$ implies that twice the vector a (with components a^i) should be an element of the coroot lattice of G . Thus this solution corresponds to the well known case of a *discrete* Wilson line [3].

The next solution we want to discuss is

$$\tilde{t} = t^{-1}. \quad (14)$$

¹Here, we have to specify that we first apply $x^5 \rightarrow -x^5$ and afterwards shift by $2\pi R$.

²A further constraint can be derived by taking the hermitian conjugate of (8) and requiring that the relations $H_i = H_i^\dagger$ and $E_\alpha^\dagger = E_{-\alpha}$ are respected. This gives the condition that t should be unitary.

This implies $\tilde{T} = -T$ and corresponds to a *continuous* Wilson line. Defining a conjugated Cartan-Weyl basis

$$\hat{H}_i = t^{1/2} H_i t^{-1/2}, \quad (15)$$

$$\hat{E}_\alpha = t^{1/2} E_\alpha t^{-1/2} \quad (16)$$

it is easy to see that (8) can be rewritten as

$$(\theta, e) : \begin{array}{ll} \hat{H}_i & \rightarrow \hat{H}_i, & i = 1, \dots, r \\ \hat{E}_{\alpha_k} & \rightarrow \exp\{2\pi\alpha_k \cdot V\} \hat{E}_{\alpha_k}, & k = 1, \dots, \dim(G) - r \end{array} \quad (17)$$

Since these relations will be important in the following let us derive them in detail. The first line of (17) is obtained as follows. First we observe that

$$(\theta, e) : \hat{H}_i = t^{1/2} H_i t^{-1/2} \rightarrow \tilde{t}^{1/2} t H_i t^{-1} \tilde{t}^{-1/2}. \quad (18)$$

Using (14) to express \tilde{t} in terms of t gives the desired result. The second line in (17) follows in a completely analogous way. Note also that the conjugation with $t^{1/2}$ takes a Cartan-Weyl basis to another Cartan-Weyl basis implying in particular that an alternative expression for the second line in (17) is

$$\hat{E}_\alpha \rightarrow e^{2\pi i V^i \hat{H}_i} \hat{E}_\alpha e^{-2\pi i V^i \hat{H}_i}. \quad (19)$$

We can now describe the geometric picture of Wilson line symmetry breaking. In all cases the projections at the fixed points are given as shift embeddings. A discrete Wilson line changes the shift embedding by replacing V with $V + a$. For the continuous Wilson line the shift embeddings at the two fixed points are both given by the same shift vector V . The difference is in the Cartan-Weyl basis the shift embedding refers to, namely H and \hat{H} , respectively. The embedding of the Cartan-Weyl basis at the fixed point $x^5 = \pi R$ is continuously rotated by the Wilson line. In particular the embedding of the Cartan torus is rotated (t cannot be generated by the H_i in the case of a continuous Wilson line). The unbroken gauge group in four dimensions is generated by elements of the Lie algebra of G which are invariant under the projections at all fixed points. For a continuous Wilson line the rank of the gauge group is reduced.

To close this section let us give a prescription on how to parameterise the general solution to (14). In this context, the term ‘general’ deserves some discussion. The Wilson line T can be also viewed as a vacuum expectation value for the internal gauge field component A_5 [3]. We do not want to count gauge equivalent vacua. In order to be a solution to (14) T has to be a linear combination of generators which are not invariant under the orbifold group. All gauge inequivalent solutions can be found by identifying a set of hermitian and mutually commuting operators C_1, \dots, C_n within the non-invariant generators. The continuous Wilson line is then a linear combination of those operators

$$T = \sum_{i=1}^n \lambda_i C_i, \quad \lambda_i \text{ real}. \quad (20)$$

One can justify this prescription by establishing the connection to the case that the orbifold is embedded as a rotation. Within the bulk gauge algebra one can identify a Cartan subalgebra containing C_1, \dots, C_n . With respect to that Cartan subalgebra the orbifold acts as a rotation. C_1, \dots, C_n are non-invariant Cartan operators. Eq. (20) corresponds to the prescription known from rotational embeddings [44, 45]. (The above argument may also be used for a systematic construction of rotational embeddings.)

For completeness, let us mention also a solution to (11) (with the upper sign) corresponding to a superposition of a continuous and a discrete Wilson line. This is given by

$$t = e^{2\pi i (T_1 + T_2)}, \quad (21)$$

with

$$T_1 = a^i H_i, \quad (22)$$

where a is an element of the coroot lattice (as in (13)) and

$$\tilde{T}_2 = -T_2. \quad (23)$$

So far, we have superposed our solutions for a discrete and a continuous Wilson line. In order, that this superposition solves (11) we need to satisfy that the discrete and continuous Wilson line commute

$$[T_1, T_2] = 0. \quad (24)$$

3 Continuous Wilson Line Breaking

The scales for symmetry breaking by the gauge embedding of orbifold twist and discrete Wilson lines is given by the compactification scale. The continuous Wilson line corresponds to a VEV along a flat direction [46, 47] and hence the breaking scale can be adjusted continuously between zero and the compactification scale. In realistic scenarios the flat direction should be lifted e.g. by SUSY breaking. This goes beyond our present investigations but we anticipate that depending on the SUSY breaking mechanism there is some choice for the breaking scale due to continuous Wilson lines. In the present section we consider the possibility that this is a scale where a Pati-Salam gauge symmetry is broken to the Standard Model gauge group. In particular we will rederive the model discussed in [45] using the technique introduced in the previous section. In [45] a six dimensional theory with gauge group $G = E_6$ was compactified on a T^2/\mathbb{Z}_2 orbifold. The E_6 was broken by the orbifold to $SO(10) \times U(1)$. A continuous Wilson line along the sixth direction was used to break this to $SU(5) \times U(1)$. Other possible breakings were given as $SO(10) \times U(1)$ to $SO(7) \times U(1)$ or $SU(4) \times U(1)$. A second discrete Wilson line along the fifth direction resulted in a further symmetry breaking (see figure 1 for a reminder). We will come back to the discrete Wilson line later and first check whether the alternative technique given in the previous section yields the same $SO(10) \times U(1)$ breaking patterns as in [45], for the continuous Wilson line.

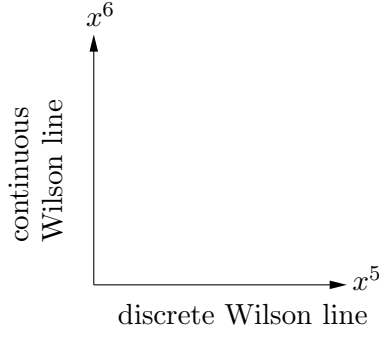


Figure 1: The setup of [45].

3.1 Gauge Symmetry Breaking

3.1.1 Breaking E_6 to $SO(10) \times U(1)$

As in [45] we embed the E_6 root vectors into an eight dimensional space by writing the six Cartan generators into the following eight dimensional vector

$$H = (H_1, H_1, H_1, H_2, H_3, H_4, H_5, H_6). \quad (25)$$

The E_6 roots are also written as eight dimensional vectors where each entry gives the eigenvalue of the root operator under the adjoint action of the Cartan generator appearing at the same position in (25). The 72 root vectors of E_6 are then

- 40 root vectors of the form

$$(0, 0, 0, \underline{\pm 1}, \underline{\pm 1}, 0, 0, 0),$$

where underlined entries can be permuted,

- 32 root vectors of the form

$$\left(\pm \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2} \right)$$

with an even number of minus signs.

In order to break E_6 to $SO(10) \times U(1)$ we use the shift vector [41]

$$V = (1, 1, 1, 0, 0, 0, 0, 0). \quad (26)$$

The spinorial root vectors listed in the second item above have half integer scalar product with V . The corresponding root operators are projected out at the fixed point $x^6 = 0$. This yields the unbroken gauge group $SO(10) \times U(1)$ where the $U(1)$ factor is generated by H_1 .

3.1.2 Continuous Wilson Line Breaking

As discussed in section 2 the effect of the continuous Wilson line is given by conjugating the Cartan-Weyl basis with $t^{1/2}$ and taking the same shift embedding with respect to the conjugated basis. This will give also $\text{SO}(10) \times \text{U}(1)$ unbroken gauge symmetry at the fixed point $x^6 = \pi R_6$. In order to find the unbroken gauge group in four dimensions we have to specify t and eliminate those group elements which do not commute with t . Our definition for the continuous Wilson line was that

$$t = e^{2\pi iT} \quad (27)$$

is mapped onto its inverse under the shift embedding, or in other words that T is a superposition of root generators corresponding to spinorial roots. In order to find a suitable parameterisation we proceed as discussed in the end of section 2. A maximal set of hermitian and commuting operators is given by³

$$C_1 = E_\delta + E_{-\delta} \ , \ C_2 = E_\gamma + E_{-\gamma}, \quad (28)$$

with

$$\delta = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \quad (29)$$

$$\gamma = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right). \quad (30)$$

Then the general continuous Wilson line can be written as

$$T = \lambda_1 (E_\delta + E_{-\delta}) + \lambda_2 (E_\gamma + E_{-\gamma}). \quad (31)$$

For completeness let us also give a set of E_6 Cartan generators with respect to which the orbifold is embedded as a rotation:

$$H_1 - H_2, \ H_3 - H_4, \ H_3 - H_5, \ H_3 - H_6, \ E_\delta + E_{-\delta}, \ E_\gamma + E_{-\gamma}. \quad (32)$$

Now we will discuss the four dimensional unbroken gauge symmetries.

Generic Case: $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_1 \neq \pm\lambda_2$

First, we take λ_1 and λ_2 to be independent and non vanishing. In this case the following $\text{SO}(10)$ generators are invariant under conjugation with t :

- four Cartan generators: $H_1 - H_2, H_3 - H_4, H_3 - H_5, H_3 - H_6,$
- 12 root generators corresponding to root vectors of the form $\pm (0, 0, 0, 0, \underline{1}, -1, 0, 0).$

³Some tentative thoughts about such operators in the context of rank reduction can be found in [49].

These generate an $SU(4) \times U(1)$ symmetry. The rank of the gauge group has been reduced by two.

Special Case: $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_1 = \lambda_2$ (or $\lambda_1 = -\lambda_2$)

In the case that $\lambda_1 = \lambda_2$ there are additional generators which are invariant under the conjugation with t . Indeed, taking for example

$$\alpha = (0, 0, 0, 0, 0, 0, -1, -1) \quad \text{and} \quad \beta = (0, 0, 0, 0, 1, 1, 0, 0) \quad (33)$$

one can show that

$$t(E_\alpha - E_\beta)t^{-1} = E_\alpha - E_\beta, \quad (34)$$

details of this computation can be found in the appendix B. Analogous combinations exist if we simultaneously permute the last four entries in α and β . This adds six generators to $SU(4) \times U(1)$ which yields a 22 dimensional gauge symmetry. This is consistent with the findings of [45] where this gauge group has been identified as $SO(7) \times U(1)$. (Since now we have also superpositions of root operators computing the Dynkin diagram would involve diagonalising the adjoint action of Cartan generators. We do not carry out this involved calculation here, the techniques are described in [45]. What one does see is the rank reduction by two since two Cartan generators have been projected out by the Wilson line.)

For $\lambda_1 = -\lambda_2$ one replaces $E_\alpha - E_\beta$ by $E_\alpha + E_\beta$ in (34). Then the rest of the discussion is the same as in the case $\lambda_1 = \lambda_2$.

Special Case: $\lambda_1 \neq 0, \lambda_2 = 0$ (or $\lambda_1 = 0, \lambda_2 \neq 0$)

For $\lambda_2 = 0$ the following $SO(10) \times U(1)$ generators are invariant under conjugation with t :

- five Cartan generators: $H_1 - H_2, H_1 - H_3, H_1 - H_4, H_1 - H_5, H_1 - H_6$,
- 20 root generators corresponding to root vectors of the form $(0, 0, 0, \underline{1, -1}, 0, 0, 0)$.

The unbroken gauge symmetry can be identified as $SU(5) \times U(1)$. The decoupled $U(1)$ is generated by

$$5H_1 - H_2 - H_3 - H_4 - H_5 - H_6. \quad (35)$$

The case $\lambda_1 = 0, \lambda_2 \neq 0$ yields the same result after some obvious sign changes have been performed.

The possible breakings due to continuous Wilson lines are summarised in table 1

choice for parameters in (31)	unbroken gauge group
$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_1 \neq \pm\lambda_2$	$SU(4) \times U(1)$
$\lambda_1 = \pm\lambda_2 \neq 0$	$SO(7) \times U(1)$
$\lambda_1 \neq 0, \lambda_2 = 0$ (or $\lambda_1 = 0, \lambda_2 \neq 0$)	$SU(5) \times U(1)$

Table 1: All symmetry breakings of $SO(10) \times U(1)$ by the continuous Wilson line.

3.1.3 The PS to SM Breaking

In order to complete the rederivation of the model in [45] we should include the fifth direction into the discussion. Along that direction we have a discrete Wilson line

$$a_5 = (1/2, 1/2, -1, 0, 0, 0, -1/2, 1/2). \quad (36)$$

The projection at the fixed point $x^5 = \pi R^5$, $x^6 = 0$ is obtained from the shift embedding with shift vector $V + a_5$. The E_6 root vectors with an integer valued scalar product with $V + a_5$ are

- 12 root vectors of the form $(0, 0, 0, \underline{\pm 1, \pm 1}, 0, 0, 0)$,
- four root vectors of the form $(0, 0, 0, 0, 0, 0, \pm 1, \pm 1)$,
- 16 root vectors of the form $(\pm (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm (\frac{1}{2}, -\frac{1}{2}))$ with an even number of minus signs.

The unbroken gauge group at the point $x^5 = \pi R_5$, $x^6 = 0$ is $SU(6) \times SU(2)$. For vanishing continuous Wilson line, the four dimensional gauge symmetry is obtained by imposing in addition integer valued scalar products with V . This removes the spinorial roots and leads to $SU(4) \times SU(2) \times SU(2) \times U(1)$, i.e. the Pati-Salam group and an extra $U(1)$. In summary, E_6 is broken to the Pati-Salam group and an extra $U(1)$ at the compactification scale.

A continuous Wilson line can be used to break that symmetry further. We focus on the x^6 direction with the continuous Wilson line

$$T = \lambda (E_\delta + E_{-\delta}), \quad (37)$$

and δ given by (29). The background fieldstrength is zero since $\delta \cdot a_5 = 0$ (the discrete and continuous Wilson lines commute). According to our general discussion the unbroken gauge group at $(0, \pi R_6)$ is $SO(10) \times U(1)$ and at $(\pi R_5, \pi R_6)$ $SU(6) \times SU(2)$. The intersection of the unbroken groups at $(0, \pi R_6)$ and $(\pi R_5, \pi R_6)$ is again $SU(4) \times SU(2) \times SU(2) \times U(1)$. The intersection of groups at the points $(0, 0)$ and $(0, \pi R_6)$ was determined in section 3.1.2 to be $SU(5) \times U(1)$. It remains to compute the intersection at $(\pi R_5, 0)$ and $(\pi R_5, \pi R_6)$ and the intersection of all fixed point groups giving the unbroken four dimensional symmetry. Let us first discuss the points $(\pi R_5, 0)$ and $(\pi R_5, \pi R_6)$. We start e.g. with the $SU(6) \times SU(2)$ at $(\pi R_5, 0)$ and keep only generators which are invariant under conjugation with t . These are:

- five Cartan generators: $H_1 - H_2, H_1 - H_3, H_1 - H_4, H_1 - H_5$ and $H_1 - H_6$,
- six root generators with root vectors of the form $(0, 0, 0, \underline{1}, -1, 0, 0, 0)$,
- two root generators of the form $(0, 0, 0, 0, 0, 0, \underline{1}, -1)$,
- four root generators with root vectors of the form $(\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \pm(\frac{1}{2}, -\frac{1}{2}))$ with an even number of minus signs.

Working out the details of the algebra one finds for the intersection of the gauge groups unbroken at $(\pi R_5, 0)$ and $(\pi R_5, \pi R_6)$ the group $SU(3) \times SU(3) \times U(1)$, where the $U(1)$ factor is generated by

$$-3H_1 - H_2 - H_3 - H_4 + 3H_5 + 3H_6. \quad (38)$$

The four dimensional unbroken gauge group can now be obtained by e.g. keeping only those $SU(3) \times SU(3) \times U(1)$ root generators whose root vectors have an integer valued scalar product with the shift vector V (26). This removes the spinorial roots, the second $SU(3)$ is broken to $SU(2) \times U(1)$ where the $U(1)$ generator is given by

$$3H_1 - H_2 - H_3 - H_4. \quad (39)$$

The unbroken gauge group in four dimensions is $SU(3) \times SU(2) \times U(1) \times U(1)$ where the two $U(1)$ s are generated by (38) and (39). Note, that the charge vectors are orthogonal

$$\begin{aligned} q_1 &= (-1, -1, -1, -1, -1, -1, 3, 3) \\ q_2 &= (1, 1, 1, -1, -1, -1, 0, 0) \end{aligned} \rightarrow q_1 \cdot q_2 = 0. \quad (40)$$

Now, we have completely rederived the picture drawn in figure 2 of [45] (see figure 2).

3.2 Splitting Bulk Matter

Bulk matter appears in complete representations of the bulk group and splits under the orbifold projections into representations of the unbroken gauge group at a particular fixed point. Our previous calculations were simple because in most cases we needed only to know whether a commutator is zero or something else. For that one does not have to determine all the structure constants but just needs to check whether two roots add up to another root which is much simpler. In order to carry over that degree of simplicity also to the computation of bulk matter projections it is useful to view gauge and matter fields as part of the adjoint representation of E_8 . Then one needs to know only whether the sum of two E_8 roots yields another E_8 root.

We view the E_6 as a factor of an $E_6 \times U(1) \times U(1)$ subgroup of E_8 . We introduce two additional Cartans into (25)

$$(H_1 + K, H_1 + L, H_1 - K - L, H_2, H_3, H_4, H_5, H_6). \quad (41)$$

The generators K and L are two additional $U(1)$ generators which are not necessarily related to symmetries. However, as will be seen shortly, charges under these additional $U(1)$ s provide a useful tool for identifying E_6 representations within the set of E_8 generators. The E_8 root vectors are

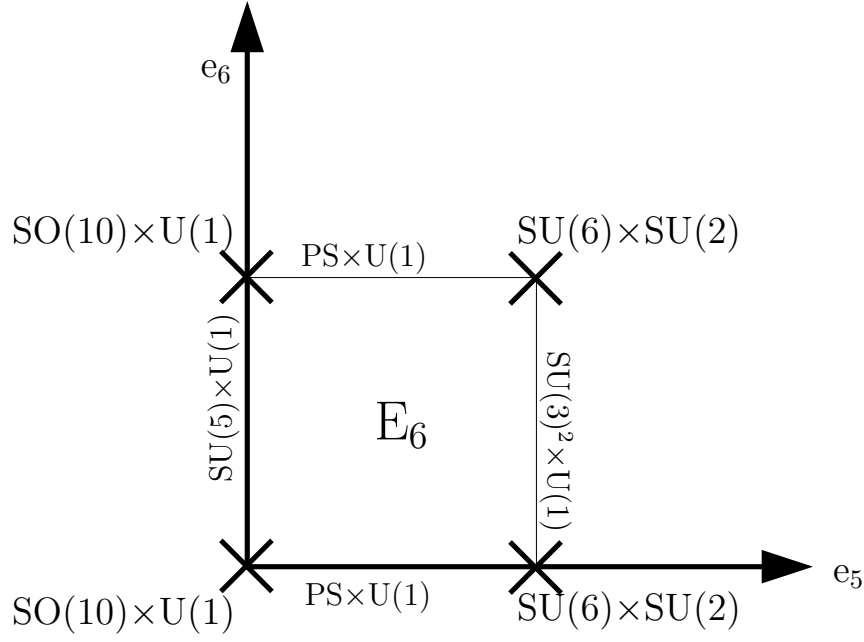


Figure 2: Local projections of the adjoint of E_6 to adjoints of groups written at the fixed points. Groups written at lines connecting two fixed points are obtained after applying both projections. The unbroken gauge symmetry in four dimensions is $SU(3) \times SU(2) \times U(1)^2$.

- 112 vectors of the form $(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)$,
- 128 vectors of the form $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$ with an even number of minus signs.

Root operators which differ from our previously used E_6 root operators fall into $E_6 \times U(1)^2$ representations. In order to find an E_6 representation we collect elements with identical K and L charges, e. g. for $K = 2/3$, $L = -1/3$ one finds 27 root generators with the following root vectors:

- one vector with $H_1 = -2/3$: $(0, -1, -1, 0, 0, 0, 0, 0)$,
- ten vectors with $H_1 = 1/3$: $(1, 0, 0, \pm 1, 0, 0, 0, 0)$,
- 16 vectors with $H_1 = -1/6$: $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$ (even number of minus signs).

That this is indeed the **27** of E_6 can be established by computing the Dynkin labels. Since a detailed discussion on how to do this can be found in the literature (e.g. [22, 50]) we refrain from the presentation of the relevant calculation. The $\overline{\mathbf{27}}$ can be obtained by multiplying the first three entries of the root vectors appearing in the **27** with -1 (reversing the (H_1, K, L) charges). There are two more pairs of **27**, $\overline{\mathbf{27}}$ in the branching of E_8 which

can be easily obtained by permuting the first three entries of our set. (The remaining six generators with root vectors of the form $(1, -1, 0, 0, 0, 0, 0)$ are E_6 singlets.)

In orbifold field theory constructions one usually chooses orbifold parities for matter by hand. These can be different for different fixed points. If we choose opposite parities for points differing only in the x^6 direction the complete E_6 multiplet will be projected out. Therefore we take identical parities for those points and it is enough to characterise the assigned parities by an ordered pair (\pm, \pm) corresponding to the values at fixed points with $x^5 = 0$, $x^5 = \pi R_5$, respectively.

After we embedded everything into the adjoint of E_8 the technical details of computing the local projections are the same as for the gauge group. We omit their presentation here, and just report the results. Further, we restrict our discussion to a **27**. The $\overline{\mathbf{27}}$ gives rise to charge conjugated states.

At the fixed point $(0, 0)$ the gauge symmetry is $SO(10) \times U(1)$ and the possible splittings of a **27** are

$$\begin{aligned} \mathbf{27} &\rightarrow \mathbf{10}_{1/3} + \mathbf{1}_{-2/3} && \text{for positive parity,} \\ \mathbf{27} &\rightarrow \overline{\mathbf{16}}_{-1/6} && \text{for negative parity.} \end{aligned} \quad (42)$$

At the fixed point $(\pi R_5, 0)$ the bulk group is broken to $SU(6) \times SU(2)$ and the **27** splits into

$$\begin{aligned} \mathbf{27} &\rightarrow (\overline{\mathbf{6}}, \mathbf{2}) && \text{for positive parity,} \\ \mathbf{27} &\rightarrow (\mathbf{15}, \mathbf{1}) && \text{for negative parity.} \end{aligned} \quad (43)$$

The intersection of gauge groups at horizontally separated fixed points is $SU(4) \times SU(2) \times SU(2) \times U(1)$. Depending on the assigned parities bulk matter splits according to

$$\begin{aligned} \mathbf{27} &\rightarrow (\mathbf{1}, \mathbf{2}, \mathbf{2})_{1/3} && \text{for } (+, +) \text{ parity,} \\ \mathbf{27} &\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_{-2/3} + (\mathbf{6}, \mathbf{1}, \mathbf{1})_{1/3} && \text{for } (+, -) \text{ parity,} \\ \mathbf{27} &\rightarrow (\overline{\mathbf{4}}, \mathbf{2}, \mathbf{1})_{-1/6} && \text{for } (-, +) \text{ parity,} \\ \mathbf{27} &\rightarrow (\mathbf{4}, \mathbf{1}, \mathbf{2})_{-1/6} && \text{for } (-, -) \text{ parity.} \end{aligned} \quad (44)$$

Following our general discussion the gauge groups and matter spectrum at vertically separated fixed point are the same but the embedding into the bulk gauge group and matter differs. For $x^5 = 0$ the intersection is $SU(5) \times U(1)$ and the local projections of the bulk **27** intersect in

$$\begin{aligned} \mathbf{27} &\rightarrow \overline{\mathbf{5}}_{8/3} && \text{for positive parity,} \\ \mathbf{27} &\rightarrow \overline{\mathbf{10}}_{-4/3} && \text{for negative parity.} \end{aligned} \quad (45)$$

Accordingly for $x^5 = \pi R_5$ the intersection of the two $SU(6) \times SU(2)$ is $SU(3) \times SU(3) \times U(1)$ and the intersections for the matter projections are

$$\begin{aligned} \mathbf{27} &\rightarrow (\mathbf{1}, \overline{\mathbf{3}})_{-2} + (\mathbf{3}, \mathbf{1})_2 && \text{for positive parity,} \\ \mathbf{27} &\rightarrow (\overline{\mathbf{3}}, \mathbf{3})_0 && \text{for negative parity.} \end{aligned} \quad (46)$$

Finally, the massless fields in four dimensions are given as the intersections from all fixed

points. One obtains the following $SU(3) \times SU(2) \times U(1) \times U(1)$ multiplets

$$\begin{aligned}
\mathbf{27} &\rightarrow (\mathbf{1}, \mathbf{2})_{-2,1} && \text{for } (+, +) \text{ parity,} \\
\mathbf{27} &\rightarrow (\bar{\mathbf{3}}, \mathbf{1})_{0,2} && \text{for } (+, -) \text{ parity,} \\
\mathbf{27} &\rightarrow (\mathbf{1}, \mathbf{1})_{-2,-2} + (\mathbf{3}, \mathbf{1})_{2,0} && \text{for } (-, +) \text{ parity,} \\
\mathbf{27} &\rightarrow (\bar{\mathbf{3}}, \mathbf{2})_{0,-1} && \text{for } (-, -) \text{ parity.}
\end{aligned} \tag{47}$$

Note, that counting the number of states surviving projections with any parity gives 15 and not 27. The continuous Wilson line removes states independent of the assigned parity. Effectively these states become massive via the coupling to an internal gauge field component which acquires a VEV. We were able to derive the symmetry breaking patterns without any explicit computation of Yukawa couplings. The reason for this is that couplings between bulk matter and the continuous Wilson line are completely fixed by higher dimensional gauge invariance.

We are now in a position to draw figure 2 for bulk matter. We do this in figure 3 with a $\mathbf{27}$ in the bulk and assigned parity $(-, -)$.

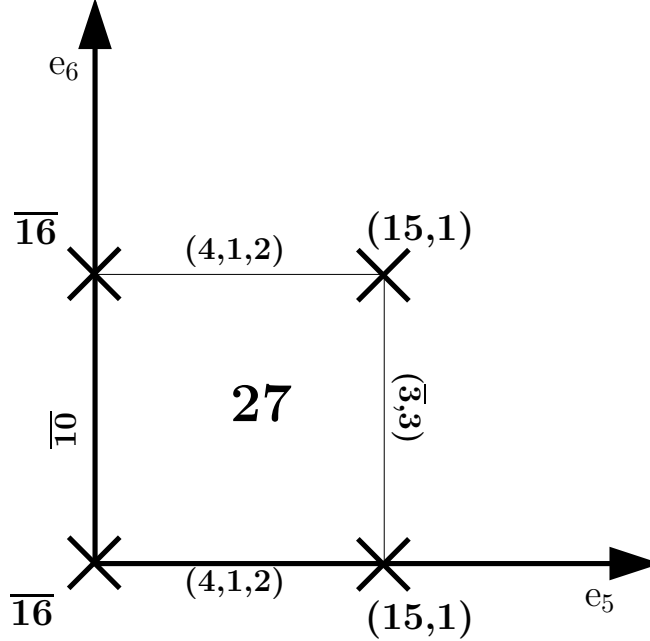


Figure 3: Local projections of the $\mathbf{27}$. At all fixed points negative parity has been assigned. States written at lines connecting two fixed points survive both projections. The massless state in four dimensions is $(\bar{\mathbf{3}}, \mathbf{2})$.

4 Electroweak Symmetry Breaking

In the rest of the paper we will explore the possibility to assign the electroweak symmetry breaking to continuous Wilson lines.

As an illustrating example we focus on models where an $SO(10)$ gauge symmetry appears at an intermediate level in the breaking of the heterotic $E_8 \times E_8$ gauge symmetry to the Standard Model symmetry in four dimensions. As argued in [39, 51] such models can be phenomenologically favourable since some of the appealing features in $SO(10)$ GUT theories are inherited even if $SO(10)$ is broken in four dimensions.

In the example of the previous section such an intermediate $SO(10)$ appeared at the origin of the extra dimensions torus lattice. This example is, however, not suitable in the context of electroweak symmetry breaking by continuous Wilson lines. The 78 dimensional adjoint representation splits into the 45 dimensional adjoint of $SO(10)$, a **16**, a $\overline{\mathbf{16}}$ and a singlet of $SO(10)$. Hence the continuous Wilson lines transform in 16 dimensional representations whereas the Standard Model Higgs should be embedded into a ten dimensional representation of $SO(10)$. From that example we learn what we have to look for. The $SO(10)$ should come from a broken group G whose adjoint branches into the **45** and **10** and possibly other $SO(10)$ representations. In such models a continuous Wilson line transforming in the **10** exists. Under the subsequent breaking to the Standard Model gauge group the **10** can split such that only an $SU(2)$ doublet remains massless. This would give rise to a continuous Wilson line corresponding to the Standard Model Higgs doublet. (In the models considered later there will be more than one **10** giving rise to more doublets in four dimensions. Among those doublets there will be always the pair of the MSSM.)

In the following two sections we will present two examples where these ideas are realised to some extent. The intermediate $SO(10)$ will arise from a \mathbb{Z}_2 breaking of $SO(14)$ and $SO(12)$. The branching rules for such breakings fall into the category discussed above.

5 A \mathbb{Z}_6 -II Example

In this section we consider a heterotic $E_8 \times E_8$ \mathbb{Z}_6 -II orbifold. The group \mathbb{Z}_6 is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2$. The embedding of \mathbb{Z}_3 will be taken such that the visible E_8 is broken to $SO(14) \times U(1)$. The subsequent \mathbb{Z}_2 breaking to $SO(10)$ will give rise to continuous Wilson lines in ten dimensional representations of $SO(10)$. A combination of discrete and continuous Wilson lines will be applied to finally break the symmetry to the colour $SU(3)$ the electromagnetic $U(1)$ and extra $U(1)$ s. Here, we assume that the three families of chiral matter originate from 16 dimensional representations of the intermediate $SO(10)$. In the following, we will separate the discussion of matter from our studies and focus on the gauge sector, only.

Before describing the details of gauge symmetry breaking we recall the structure of the six dimensional space the heterotic string is compactified on (see e.g. [40]). This is taken as a \mathbb{Z}_6 -II orbifold of a six dimensional torus. The six dimensional torus is a product of three two dimensional tori whose compactification lattices are the root lattices of G_2 , $SU(3)$ and

$SO(4)$, respectively. In each of the torus lattices the orbifold acts as a rotation. The action of \mathbb{Z}_6 is characterised by a three component vector whose entries are rotation angles in 2π units. For \mathbb{Z}_6 -II this vector is

$$v = \left(\frac{1}{6}, \frac{1}{3}, -\frac{1}{2} \right). \quad (48)$$

One characteristic feature of \mathbb{Z}_6 -II is that \mathbb{Z}_3 as well as \mathbb{Z}_2 posses fixed tori. The structure of the compact space is visualised in figure 4.

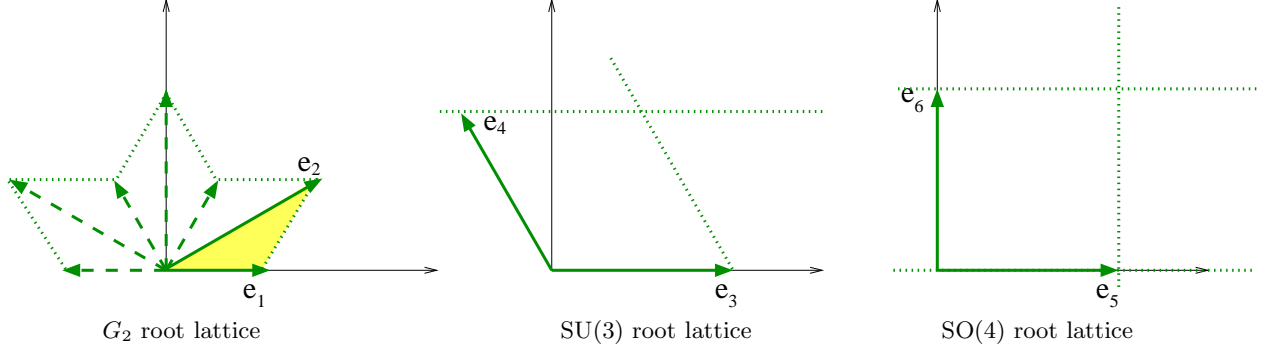


Figure 4: Geometry of compact space.

5.1 Rotational Embedding

In this section we discuss the model within the rotational embedding approach of [44]. We will focus only on the first E_8 . In the next section, we will consider an equivalent model where the orbifold is embedded as a shift. For the equivalent model we will also give the components of shift vectors and discrete Wilson lines within the second E_8 such that modular invariance is ensured.

The orbifold group is $\mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$. We embed the \mathbb{Z}_3 factor as a shift and the \mathbb{Z}_2 factor as a rotation. The \mathbb{Z}_3 shift is taken to be⁴

$$V_3 = \left(\frac{4}{3}, 0^7 \right), \quad (49)$$

where 0^n is shorthand for n vanishing consecutive components. E_8 roots having integer valued scalar product with V_3 are

$$(0, \underline{\pm 1, \pm 1, 0^5}). \quad (50)$$

Hence, \mathbb{Z}_3 breaks E_8 to $SO(14) \times U(1)$. The \mathbb{Z}_2 is embedded as a rotation acting on the 16 extra bosonic chiral fields X^I of heterotic string theory which are compactified on an

⁴Compared to (41) we use the canonical E_8 Cartan vector $(H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8)$ in the rest of the paper.

$E_8 \times E_8$ lattice. This induces a rotation acting on Cartan vectors ($\sim \partial X^I$). Root vectors are charge vectors for the adjoint representation and are rotated accordingly. Consistency requires that root vectors are mapped to root vectors or in other words that the rotation is an automorphism of the $E_8 \times E_8$ root lattice. For details see [44, 45]. We take the following $SO(8)$ matrix for the rotational embedding of \mathbb{Z}_2 within the first E_8

$$s = \text{diag}(1, 1, 1, 1, -1, -1, -1, -1). \quad (51)$$

Actually, it is not enough to define an automorphism on the root lattice. Indeed, one should specify an automorphism on the E_8 Lie algebra. We will come back to that later and for the time being keep the discussion at a schematic level. For now, we will also just compute the rank and dimension of unbroken gauge groups and guess the algebra. All these results have been computed in a rigorous way by realising the gauge embedding as an algebra automorphism. For presentational purposes the sketchy derivations given at the moment are more suitable.

The Cartan operators which are invariant under a rotation by s are

$$H_1, \dots, H_4. \quad (52)$$

There are 12 invariant roots of $SO(14)$ of the form

$$(0, \pm 1, \pm 1, 0, 0^4).$$

For the remaining 72 roots a superposition of a root operator with its image is invariant. That leads to 36 more generators for the unbroken gauge group. So, altogether the unbroken gauge group has dimension 52. The orbifold alone does not reduce the rank of the gauge group. The four Cartan generators which have been projected out can be replaced by the invariant combinations

$$E_{\gamma_i} + E_{-\gamma_i}, \quad i = 1, \dots, 4 \quad (53)$$

with

$$\gamma_1 = (0^4, 1, 1, 0, 0), \quad \gamma_2 = (0^4, 1, -1, 0, 0), \quad \gamma_3 = (0^6, 1, 1), \quad \gamma_4 = (0^6, 1, -1). \quad (54)$$

Indeed, these operators mutually commute ($\gamma_i \pm \gamma_j$ is not an E_8 root) and are neutral under the adjoint action with the four Cartans in (52). So, the unbroken gauge group has rank 8 and dimension 52 which fits with⁵

$$SO(10) \times SU(2) \times SU(2) \times U(1), \quad (55)$$

where the extra $U(1)$ factor is the same as in the previous \mathbb{Z}_3 breaking of E_8 to $SO(14) \times U(1)$. The \mathbb{Z}_2 breaking of $SO(14)$ to $SO(10)$ falls into the class of models considered in the previous section. Hence, continuous Wilson lines of the form

$$W = \sum_{i=5}^8 \lambda_i H_i \quad (56)$$

⁵The rotational embedding (51) breaks E_8 to $E_7 \times SU(2)$ [50].

are of potential interest for electroweak symmetry breaking. These Wilson lines can correspond to shifts by e_5 or e_6 . (For details on continuous Wilson lines in rotational embeddings see [44, 45].) Before we study the continuous Wilson line breaking we will break $\text{SO}(10)$ to the Standard Model gauge group with discrete Wilson lines.

First, we introduce \mathbb{Z}_3 discrete Wilson lines. Shifts along e_3 and e_4 will be associated with a shift in E_8 . The shift vector is chosen as

$$a_3 = a_4 = \left(0, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, 0^4\right). \quad (57)$$

In the presence of that Wilson line, the \mathbb{Z}_3 sector of the orbifold breaks more gauge symmetry, E_8 root vectors having integer valued scalar products with V_3 and a_3 are of the form

$$\begin{aligned} (0, \underline{1, -1, 0}, 0^4) &\rightarrow \text{SU}(3), \\ (0^4, \underline{\pm 1, \pm 1, 0}, 0) &\rightarrow \text{SO}(8), \end{aligned} \quad (58)$$

and the previously unbroken $\text{SO}(14) \times \text{U}(1)$ is broken further to

$$\text{SU}(3) \times \text{SO}(8) \times \text{U}(1)^2 \quad (59)$$

by the Wilson line (57). Breaking this further down by the \mathbb{Z}_2 orbifold leads to $\text{SU}(3) \times \text{SU}(2)^4 \times \text{U}(1)^2$ unbroken symmetry in four dimensions.

We can obtain a smaller four dimensional gauge group by introducing an additional \mathbb{Z}_2 discrete Wilson line. To this end, we associate shifts by e_5 with a rotation in the E_8 root lattice. That rotation describes a discrete Wilson line if it commutes with the orbifold rotation s in (51). We take

$$a_5 = \text{diag}(1, 1, 1, 1, -\sigma, -\sigma), \quad \text{with } \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (60)$$

To our knowledge, this option of describing discrete Wilson lines in rotationally embedded orbifolds has not been considered in the literature, so far. Later we will argue that in a basis where the orbifold becomes shift embedded this Wilson line corresponds to a discrete Wilson line as introduced in [3].

Generators for the unbroken gauge group in four dimensions are obtained by imposing invariance under s and a_5 rotations on the generators of $\text{SU}(3) \times \text{SU}(2)^4 \times \text{U}(1)^2$. Again, the four Cartan operators (52) are invariant. The six $\text{SU}(3)$ roots in (58) are also invariant under orbifold and Wilson line. The structure of invariant superpositions of root operators is a bit more complex than previously. Some orbifold invariant superpositions are also invariant under the Wilson line. This happens if either the involved roots are invariant under the rotation by a_5 or the s and a_5 image of a root are identical. This applies to the four new Cartan generators in (53) and we see that the discrete Wilson line does not reduce the rank of the gauge group. The remaining $24 - 8 = 16$ roots of $\text{SO}(8)$ lead to four invariant superpositions of four root generators. Altogether, the rank of the unbroken gauge group is eight and the dimension is 18. This fits with

$$\text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \times \text{SU}(2) \times \text{U}(1)^3.$$

This result has been confirmed by analysing the algebra of the unbroken gauge group as we will discuss below. Such an analysis shows also that an $SU(3) \times SU(2) \times U(1)$ factor is completely embedded into the intermediate $SO(10)$ in (55). We interpret that subgroup as the Standard Model group.

Let us now study the symmetry breaking by a continuous Wilson line. The continuous Wilson line we are going to consider appears as a non trivial embedding of a lattice shift by e_6 . The embedding of the discrete Wilson line as a rotation provides a novel feature, *viz.* the stabilisation of some parameters in the continuous Wilson line (56). Shifts by e_5 and e_6 should commute and hence the continuous Wilson line can only point into directions which are left invariant under the rotation a_5 in (60). That reduces the number of four parameters in (56) to two

$$W = \lambda (H_5 - H_6) + \lambda' (H_7 - H_8). \quad (61)$$

The effective four dimensional picture for such a moduli stabilisation is that a non vanishing fieldstrength F_{56} results in mass terms for some of the continuous Wilson line moduli. Since the stabilisation mechanism is due to a discrete Wilson line the associated scale is the compactification scale.

Let us focus on the example $\lambda \neq 0$ and $\lambda' = 0$. In this case, generators which are charged under $H_5 - H_6$ are projected out. This removes the Cartan operator $E_{\gamma_2} + E_{-\gamma_2}$ from the list in (53) and the rank is reduced by one. All quadruplets forming superposition invariant under s and a_5 are projected out as well. Hence, the unbroken gauge group is

$$SU(3) \times U(1)^5. \quad (62)$$

That this breaking corresponds indeed to electroweak symmetry breaking as far as the Standard Model subgroup of the intermediate $SO(10)$ is concerned can be confirmed by a detailed analysis of the Lie algebras. The distribution of all appearing Wilson lines within the compact space is summarised in figure 5.

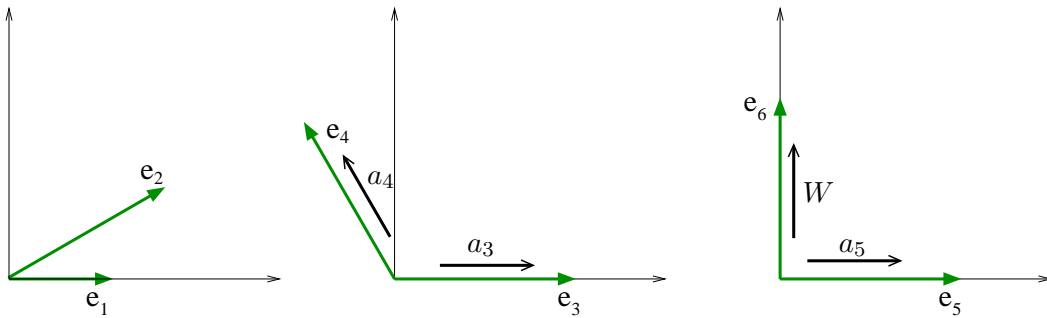


Figure 5: The setup of the model. The Wilson lines a_3 , a_4 , and a_5 are *discrete*, where the first two, a_3 and a_4 , are realised as shifts, and the latter, a_5 , as a rotation. W is *continuous*. The geometry does not allow for Wilson lines in the first torus.

After we presented the symmetry breaking pattern at a schematic level we put the derivations on firm ground now. To this end, we need to provide the orbifold embedding as an algebra automorphism and work out the algebras of the unbroken gauge symmetries. We employ the same techniques as we did in [45] where also more details can be found.

So far the \mathbb{Z}_2 orbifold and Wilson line embeddings have been given as automorphisms of the E_8 root lattice in (51) and (60), respectively. As such they can be written as a product of Weyl reflections. The orbifold embedding (51) can be obtained as the product of four Weyl reflections at γ_1 to γ_4 in (54). Methods on how to obtain an algebra automorphism from Weyl reflections are discussed e.g. in [52, 53]. A canonical guess for the algebra automorphism belonging to the orbifold action (51) is the conjugation with

$$e^{\frac{i\pi}{2} \sum_{j=1}^4 (E_{\gamma_j} + E_{-\gamma_j})}, \quad (63)$$

where the appearing operators are the same as given in (53). Next, one needs to ensure that the algebra automorphism represents a \mathbb{Z}_2 . All structure constants of E_8 have to be determined. A systematic way of doing that can be found in [54], see also Appendix B. We used a computer programme to perform that calculation and found that the algebra automorphism is indeed \mathbb{Z}_2 . It corresponds to the lift of conjugacy class 44 in [50].

The rotation (60) belonging to the discrete Wilson line can be written as the product of Weyl reflections at γ_1 and γ_3 . Here, the situation is slightly more complicated since the canonical guess provides a \mathbb{Z}_4 automorphism of E_8 . As in [45], we supplement the canonical guess with an additional shift. Embedding the Wilson line (60) as a conjugation with

$$e^{\frac{i\pi}{2} \sum_{j=1}^4 H_j} e^{\frac{i\pi}{2} (E_{\gamma_1} + E_{-\gamma_1} - E_{\gamma_3} - E_{-\gamma_3})} \quad (64)$$

provides a consistent \mathbb{Z}_2 representation. The additional minus sign in front of $E_{\pm\gamma_3}$ is not important for the consistency of the algebra automorphism but for modular invariance. We will comment on that shortly.

After the complete embedding into the algebra has been fixed the unbroken gauge groups can be identified also on an algebraic level. We have checked that our previous claims on the gauge symmetry breaking pattern are correct. The detailed calculations go along the same lines as the one presented in [45]. Instead of reporting them here we will present an equivalent model where orbifold and discrete Wilson lines are embedded as shifts. That model will be investigated in detail in the next section by using the technique developed in section 2.

Since all generators appearing in the exponents in (63) and (64) are elements of the new Cartan algebra defined in (53) the conjugation with these elements can be associated with a shift embedding w.r.t. the new choice of a Cartan subalgebra⁶. In order to make contact to standard notation one has to perform redefinitions within the set of new Cartan generators such that the E_8 roots take their standard form given in section 3.2. First,

⁶The fact that discrete Wilson line and orbifold appear as shifts w.r.t. the same Cartan subalgebra is a direct consequence of the defining property that orbifold embedding and lattice shift embedding commute for discrete Wilson lines.

one has to diagonalise the adjoint action of the operators (53), or in other words find a Cartan Weyl basis in which the operators (53) together with H_1, \dots, H_4 form the Cartan subalgebra. This has been done with the help of a computer programme. If the eigenvalues corresponded to the standard roots of E_8 the shift vector could be read off from (63) as $(0^4, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. The actual calculation shows however, that in order to obtain the E_8 roots in their standard form one has to take e.g. the following Cartan subalgebra (denoted by $\overline{H}_1, \dots, \overline{H}_8$)

$$\begin{aligned} H_i &= \overline{H}_i \text{ for } i = 1, \dots, 4, \\ E_{\gamma_1} + E_{-\gamma_1} &= \overline{H}_8 - \overline{H}_6, \\ E_{\gamma_2} + E_{-\gamma_2} &= \overline{H}_8 + \overline{H}_6, \\ E_{\gamma_3} + E_{-\gamma_3} &= \overline{H}_7 - \overline{H}_5, \\ E_{\gamma_4} + E_{-\gamma_4} &= \overline{H}_7 + \overline{H}_5. \end{aligned} \tag{65}$$

Plugging these redefinitions into (63) and (64) one obtains expressions of the form

$$e^{2\pi i V \cdot \overline{H}}. \tag{66}$$

For the orbifold the shift vector V is

$$V_2 = \left(0^6, \frac{1}{2}, \frac{1}{2}\right). \tag{67}$$

The Wilson line a_5 becomes also shift embedded, and slightly abusing the notation we call the shift vector a_5 , too. Taking into account the additional first term in (64) we find

$$a_5 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right). \tag{68}$$

For modular invariance it is necessary that the scalar product of V_2 with a_5 is an integer multiple of $1/2$. With our sign choice in (64) that goal has been achieved. (Another possibility is a suitable choice of hidden sector components.)

5.2 Shift Embedding

We consider a heterotic \mathbb{Z}_6 -II orbifold which is embedded into the gauge group $E_8 \times E_8$ as a shift with shift vector

$$V_6 = \left(\frac{2}{3}, 0^5, \frac{1}{2}, \frac{1}{2}\right) \left(\frac{1}{3}, 0^7\right). \tag{69}$$

Most of the time we will focus on the first E_8 only. The embedding into the second E_8 is only important in the context of modular invariance. The \mathbb{Z}_3 subgroup of \mathbb{Z}_6 is embedded with a shift vector equivalent to

$$V_3 = 2V_6 = \left(\frac{4}{3}, 0^7\right). \tag{70}$$

Within the first E_8 , the roots

$$(0, \underline{\pm 1, \pm 1}, 0^5) \quad (71)$$

have an integer valued scalar product with V_3 . Hence, the \mathbb{Z}_3 orbifold breaks the first E_8 to $SO(14) \times U(1)$. We will see now that the embedding of the \mathbb{Z}_2 subgroup of \mathbb{Z}_6 yields $SO(10)$ and continuous Wilson lines in ten dimensional representations. Indeed, the embedding of \mathbb{Z}_2 is specified by the shift vector

$$V_2 = 3V_6 = \left(0^6, \frac{1}{2}, \frac{1}{2}\right). \quad (72)$$

Roots having integer valued scalar products with V_3 and V_2 (and thus with V_6) are of the form

$$(0, \underline{\pm 1, \pm 1}, 0^3, 0, 0) \quad SO(10) \text{ roots}, \quad (73)$$

$$(0^6, \pm 1, \pm 1) \quad SU(2) \times SU(2) \text{ roots}. \quad (74)$$

The continuous Wilson lines which are of potential interest in the context of electroweak symmetry breaking come from root operators whose roots have half integer valued scalar product with V_2 . That applies to the following roots

$$(0, \underline{\pm 1, 0, 0, 0, 0}, \underline{\pm 1, 0}), \quad (75)$$

leading to a $(\mathbf{10}, \mathbf{2}, \mathbf{2})$ representation of $SO(10) \times SU(2)^2$.

In order to hierarchically separate the GUT breaking from the electroweak symmetry breaking we will project the $SO(10)$ symmetry further down to the Standard Model symmetry at the compactification scale. This can be done by introducing discrete Wilson lines. Along the lattice vectors of the $SU(3)$ torus we switch on two identical Wilson lines

$$a_3 = a_4 = \left(0, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, 0^4\right) (0^8). \quad (76)$$

This Wilson line breaks our $SO(10)$ group to $SU(3) \times SU(2) \times SU(2) \times U(1)$. In order, to obtain the Standard Model group from $SO(10)$ we need another discrete Wilson line which corresponds to one of the $SO(4)$ lattice vectors, e.g. e_5

$$a_5 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right) \left(0^6, \frac{1}{2}, \frac{1}{2}\right). \quad (77)$$

This finally, breaks $SO(10)$ to $SU(3) \times SU(2) \times U(1) \times U(1)$, where the roots are given by

$$\begin{aligned} \pm (0^4, 1, 1, 0, 0) &\rightarrow SU(2), \\ \pm (0, \underline{1, -1}, 0, 0^4) &\rightarrow SU(3). \end{aligned} \quad (78)$$

The decoupled $U(1)$ s (with $SO(10)$ embedding) are generated by

$$H_5 - H_6 \text{ and } H_2 + H_3 + H_4. \quad (79)$$

In particular the hypercharge corresponds to (see e.g. [55])

$$Y_W = \frac{2}{3} (H_2 + H_3 + H_4) - (H_5 - H_6). \quad (80)$$

Now, let us have a closer look at continuous Wilson lines. These will be associated to shifts with the $SO(4)$ lattice vector e_6 . According to our general rules from section 2 we have to find a maximal set of mutually commuting hermitian operators built from root operators corresponding to the roots in (75). These are given by

$$E_{\gamma_i} + E_{-\gamma_i}, \quad i = 1, \dots, 4, \quad (81)$$

with

$$\gamma_1 = (0^4, 1, 0, 1, 0), \quad \gamma_2 = (0^4, 1, 0, -1, 0), \quad \gamma_3 = (0^5, 1, 0, 1), \quad \gamma_4 = (0^5, 1, 0, -1). \quad (82)$$

In the presence of discrete Wilson lines we can switch on only continuous Wilson lines which commute with the discrete ones. This leaves only the operators corresponding to γ_1 and γ_3 . Consider for instance

$$T = \lambda (E_{\gamma_1} + E_{-\gamma_1}). \quad (83)$$

This Wilson line breaks the $SU(2)$ in (78) times the $U(1)$ generated by $H_5 - H_6$ (see (79)) to a $U(1)$ with generator H_6 . The $SU(3)$ and the other $U(1)$ in (79) commute with the continuous Wilson line. Identifying the weak isospin as the $U(1)$ generator of the broken $SU(2)$

$$I_W^3 = \frac{1}{2} (H_5 + H_6) \quad (84)$$

we find that the electromagnetic $U(1)$

$$Q_{em} = I_W^3 + Y_W/2 = \frac{1}{3} (H_2 + H_3 + H_4) + H_6 \quad (85)$$

is left unbroken. In addition, there will be another unbroken $U(1)$ which is embedded into $SO(10)$

$$H_2 + H_3 + H_4 - H_6. \quad (86)$$

That this extra $U(1)$ appears is a consequence of the construction breaking the $SO(10)$ by discrete Wilson lines to the Standard Model group. Since discrete Wilson lines do not reduce the rank there will be always one additional $U(1)$ at that stage. The rank reducing property of our Wilson line (83) was used to lower the rank of isospin times hypercharge by one.

So far, we have focused on the breaking pattern within the intermediate $SO(10)$. Here, the considered continuous Wilson line has indeed the features of a standard model Higgs (or more precisely the MSSM pair, E_{γ_1} and $E_{-\gamma_1}$ carry opposite hypercharges). If we come back to the string model we started with, there are some caveats to be mentioned. Before we switched on the discrete Wilson lines breaking $SO(10)$ to the SM group there were in addition to $SO(10)$ also two unbroken $SU(2)$ groups (74). One of these $SU(2)$ s is broken by the discrete Wilson lines and the other one by the continuous Wilson line. The picture can look different in other models as will be seen in the next section.

6 A $\mathbb{Z}_2 \times \mathbb{Z}_2$ Example

As the compact space we take the product of three $\text{SO}(4)$ lattices (figure 6) modded by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ act as rotations defined by the vectors (see

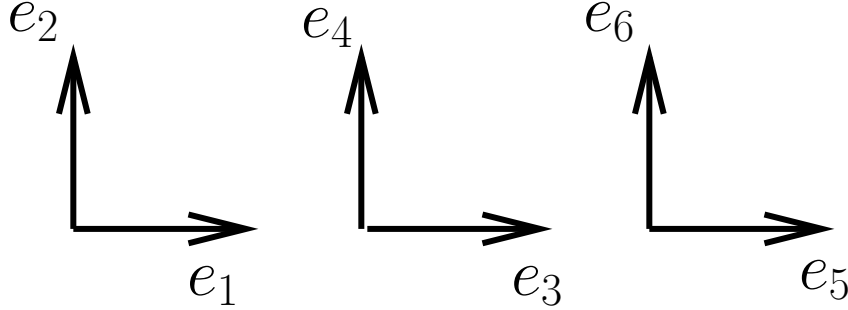


Figure 6: The six dimensional compact space.

discussion around (48) for an explanation of the notation)

$$v_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0 \right) \quad , \quad v_2 = \left(0, \frac{1}{2}, -\frac{1}{2} \right). \quad (87)$$

For more details of the geometry and the fixed point structure see e.g. [39].

Before going into details of the model we first outline the rough picture. The embedding of the orbifold into the gauge group $E_8 \times E_8$ will be such that one of the \mathbb{Z}_2 breaks the visible E_8 to a subgroup containing $\text{SO}(12)$. The other generator breaks the $\text{SO}(12)$ to $\text{SO}(10)$. This falls into the general pattern discussed in section 4 and thus gives rise to continuous Wilson lines in ten dimensional representations of $\text{SO}(10)$. Discrete Wilson lines will break the $\text{SO}(10)$ down to the Standard Model group with an extra $\text{U}(1)$ and reduce the continuous Wilson lines to $\text{SU}(2)$ doublets.

6.1 Breaking E_8 to SM via $\text{SO}(10)$

Now, we provide the details of the construction. Again, we consider the hidden E_8 only when it is important for satisfying modular invariance conditions. The \mathbb{Z}_2 generated by v_1 is shift embedded into $E_8 \times E_8$ with

$$V_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0^6 \right) (1, 0^7), \quad (88)$$

breaking $E_8 \times E'_8$ to $E_7 \times \text{SU}(2) \times \text{SO}(16)'$. The orbifold twist acts as 180° rotations in the first two tori in figure 6. Within these two tori we also introduce a discrete Wilson line which we associate with shifts along e_1

$$a_1 = (1, 0^7) \left(0^2, \frac{1}{2}, \frac{1}{2}, 0^2, \frac{1}{2}, -\frac{1}{2} \right). \quad (89)$$

The last torus (spanned by e_5 and e_6) is invariant under the actions of v_1 and a_1 . The roots of operators belonging to the gauge symmetry in the bulk of the corresponding orbifold have integer valued scalar products with V_1 and a_1 . Within the first E_8 this applies to

$$\begin{aligned} (\pm 1, \pm 1, 0^6) &\rightarrow \text{SU}(2)^2, \\ (0, 0, \pm 1, \pm 1, 0, 0, 0, 0) &\rightarrow \text{SO}(12). \end{aligned} \quad (90)$$

Hence the gauge group in the bulk of the third lattice in figure 6 is

$$\text{SO}(12) \times \text{SU}(2) \times \text{SU}(2). \quad (91)$$

The next step is to break that gauge group to an $\text{SO}(10)$ intermediate group. This can be achieved by embedding the $\mathbb{Z}_2 \times \mathbb{Z}_2$ generator v_2 with a shift

$$V_2 = \left(0, \frac{1}{2}, -\frac{1}{2}, 0^5\right) (0^8). \quad (92)$$

Indeed, $\text{SO}(12) \times \text{SU}(2)^2$ roots (90) having integer valued scalar product with V_2 are

$$(0^3, \pm 1, \pm 1, 0, 0, 0) \rightarrow \text{SO}(10). \quad (93)$$

In contrast to our previous example the extra $\text{SU}(2)$ factors in (90) are now broken at the compactification scale. The continuous Wilson lines which are of potential interest for electroweak symmetry breaking correspond to $\text{SO}(12)$ roots having half integer scalar product with V_2

$$(0, 0, \pm 1, \pm 1, 0, 0, 0, 0). \quad (94)$$

These form two **10** representations of $\text{SO}(10)$.

In the following we will break $\text{SO}(10)$ further down to the Standard Model group by means of discrete Wilson lines. We will discuss two setups with different gauge group geographies in the extra dimensions and identical pictures in four dimensions. In the first model we introduce two discrete Wilson lines within the third torus

$$\begin{aligned} a_5 &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \left(0, \frac{1}{2}, -\frac{1}{2}, 0^5\right), \\ a_6 &= \left(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right). \end{aligned} \quad (95)$$

$\text{SO}(10)$ roots having integer valued scalar products with these two Wilson lines are

$$\begin{aligned} (0^3, 1, -1, 0, 0, 0) &\rightarrow \text{SU}(3), \\ (0^6, \underline{1, -1}) &\rightarrow \text{SU}(2), \end{aligned} \quad (96)$$

and hence we succeeded to obtain a four dimensional Standard Model group which is embedded into an intermediate $\text{SO}(10)$. Among the additional $\text{U}(1)$ factors one combination corresponds to the hypercharge

$$Y_W = \frac{2}{3} (H_4 + H_5 + H_6) - H_7 - H_8. \quad (97)$$

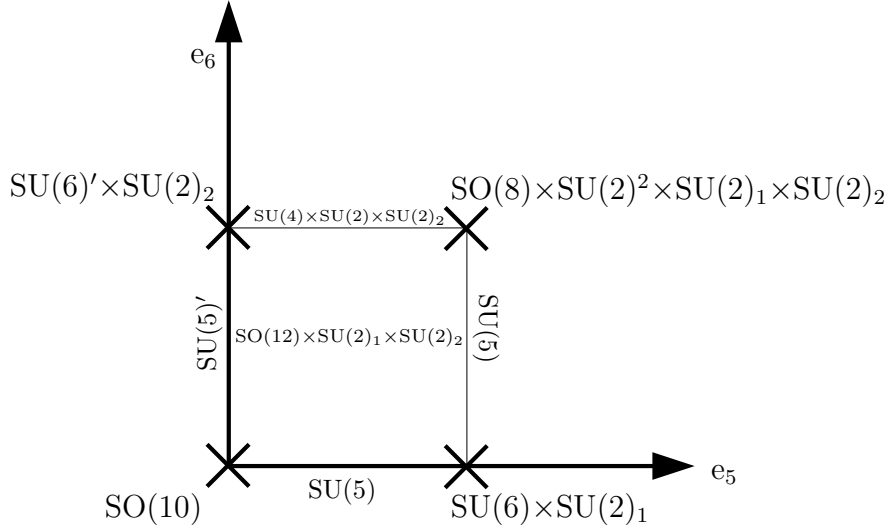


Figure 7: Local projections in the third torus. $U(1)$ factors are suppressed. A prime indicates a different embedding into the bulk group. Groups written at lines connecting two fixed points correspond to simultaneously imposing the projection conditions of both fixed points. The gauge symmetry in four dimensions is $SU(3) \times SU(2)$.

Zooming in on the third torus in figure 6 we obtain the discussed breaking in figure 7.

The second setup we want to consider is closely related to the one discussed so far. We merely associate the Wilson line a_5 to the cycle generated by e_3

$$a_5 \rightarrow a_3. \quad (98)$$

The only change is a modification of figure 7 since now the Wilson line a_3 breaks the bulk group in the depicted torus. The modified picture is shown in figure 8.

6.2 Electroweak Breaking

Finally, we turn on a continuous Wilson line. This will be associated with lattice shifts by e_5 , i.e. we superpose the discrete Wilson line a_5 (95) with a continuous Wilson line. From root operators belonging to the set (94) we form a maximal set of hermitian and mutually commuting operators. In addition, these operators should commute with all discrete Wilson lines. In other words, we consider only a subset of roots in (94) which have vanishing scalar products with the discrete Wilson lines. For a generic continuous Wilson line one finds

$$T = \lambda_1 C_1 + \lambda_2 C_2. \quad (99)$$

with

$$C_i = E_{\gamma_i} + E_{-\gamma_i} \quad i = 1, 2. \quad (100)$$

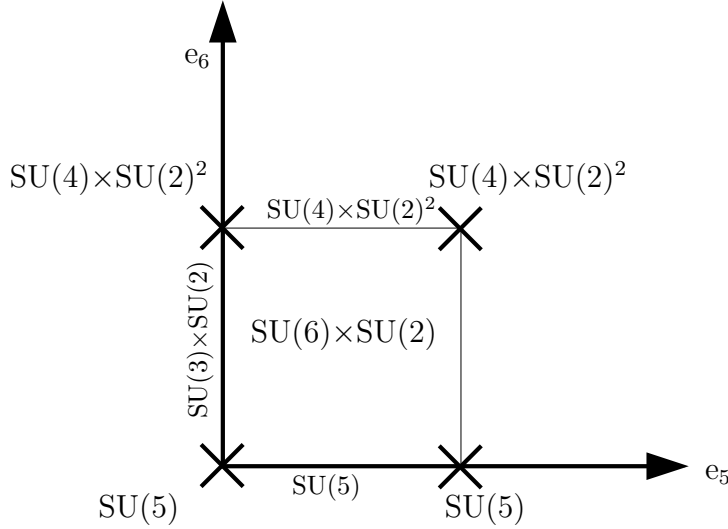


Figure 8: Figure 7 changes into the present figure if the Wilson line a_5 is moved to the direction e_3 . The spectrum at fixed points separated along the horizontal direction is identical.

and

$$\gamma_1 = (0^2, 1, 0^3, -1, 0) \quad , \quad \gamma_2 = (0^2, 1, 0^4, -1) \quad . \quad (101)$$

As an example we consider the case $\lambda_2 = 0$. The following seven Cartan generators of E_8 commute with the continuous Wilson line

$$H_1, \quad H_2, \quad H_3 + H_7, \quad H_4, \quad H_5, \quad H_6, \quad H_8. \quad (102)$$

The last five generators are embedded into the intermediate $SO(10)$ (93). The combinations $H_4 - H_5$ and $H_5 - H_6$ form the Cartan algebra of the $SU(3)$ colour which is unbroken also after the continuous Wilson line has been turned on (see (96)). The generator $H_7 - H_8$ is the Cartan generator of the $SU(2)$ in (96) and hence its charges correspond to the weak isospin. This $SU(2)$ is completely broken by the continuous Wilson line. The hypercharge generator (97) does not commute with T , either. However, the combination of isospin and hypercharge generating the electromagnetic $U(1)$ is left unbroken

$$Q_{em} = I_W^3 + Y_W/2 = \frac{1}{2}(H_7 - H_8) + Y_W/2 = \frac{1}{3}(H_4 + H_5 + H_6) - H_8. \quad (103)$$

Thus our continuous Wilson line breaking produces exactly the electroweak breaking of the MSSM. Focusing on the third torus in figure 6 we draw a geometric picture of the electroweak breaking in figure 9.

In the alternative setup of figure 8 we do not need to superpose a continuous with a discrete Wilson line. From our previous discussion only the geometric picture drawn in figure 9 is altered into the picture drawn in figure 10.

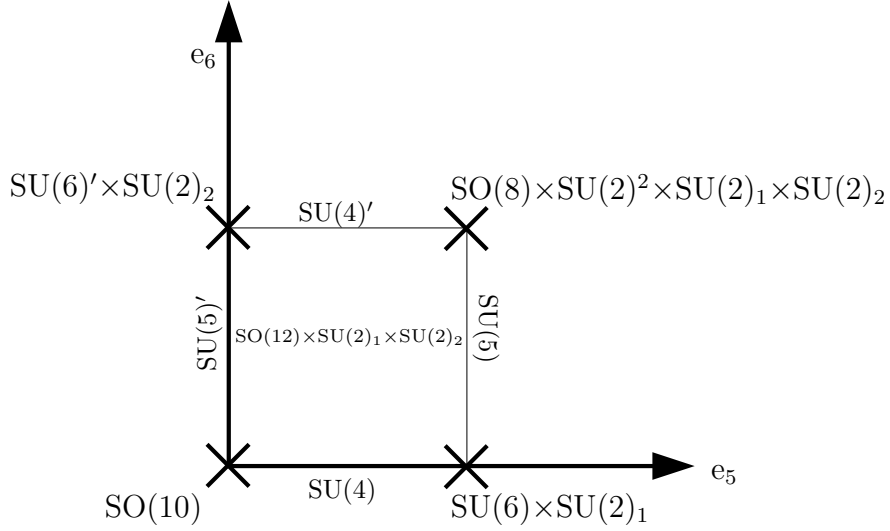


Figure 9: Compared to figure 7 a continuous Wilson line is added in the e_5 direction. The unbroken gauge group in four dimensions is $SU(3) \times U(1)^5$.

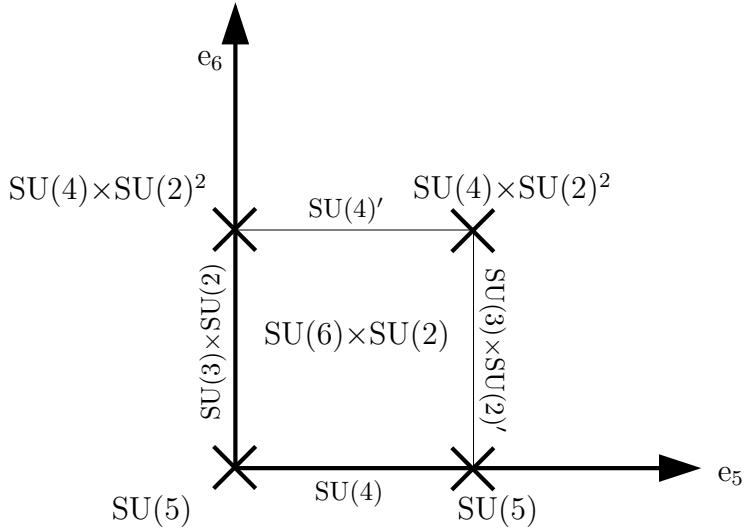


Figure 10: The continuous Wilson line lifts the degeneracy of horizontally separated fixed points.

In summary, we have achieved some improvements compared to the model discussed in section 5. Within the visible E_8 all non Abelian gauge symmetries are broken to the Standard Model symmetry at the compactification scale. The continuous Wilson line just breaks

$$SU(2)_W \times U(1)_Y \rightarrow U(1)_{em}. \quad (104)$$

7 Conclusions

In the present paper we investigated symmetry breaking due to continuous Wilson lines. Previously the mechanism of continuous Wilson line breaking had been introduced in models where the orbifold is embedded as a rotation. We were able to rephrase that technique such that it works also in shift embedded orbifolds. The geometric picture of [45] is confirmed in the new framework. Like a discrete Wilson line the continuous Wilson line also lifts the degeneracy between fixed points. The unbroken gauge symmetry, however, is the same at each fixed point. The continuous Wilson line is responsible for a misalignment of the embedding into the bulk group.

As a first application we studied the model of [45] reformulated as a shift embedded orbifold. Results of [45] are confirmed and moreover the splitting of bulk matter is given. The effective four dimensional picture corresponds to the Higgs mechanism where matter becomes massive due to Yukawa couplings to the Higgs field. For bulk matter we did not have to compute these Yukawa couplings explicitly since they are fixed by higher dimensional gauge invariance.

In our models with unbroken $N = 1$ supersymmetry the scale for symmetry breaking by continuous Wilson lines is associated to a flat direction in moduli space [46, 47]. It can be anywhere between zero and the compactification scale. (In the decompactification limit higher dimensional gauge symmetry is restored and the continuous Wilson line becomes a higher dimensional gauge degree of freedom.) In realistic models with broken supersymmetry that flat direction should be lifted and the breaking scale will be fixed. We leave the investigation of such a mechanism for future work. In the present paper, we studied the possibility that the breaking scale is given by the electroweak scale as far as the gauge symmetry breaking patterns are concerned. In a class of models where the Standard Model group appears as a subgroup of some intermediate $SO(10)$ we developed a strategy how to obtain electroweak symmetry breaking from continuous Wilson line breaking. The intermediate $SO(10)$ originates from the breaking of a larger group. This is typically the case in heterotic orbifolds. Under the orbifold breaking to $SO(10)$ the adjoint of the higher dimensional group should branch into the adjoint of $SO(10)$ and ten dimensional representations. In the subsequent breaking a Higgs doublet can be obtained from the **10** of $SO(10)$. Such a Higgs doublet corresponds to a continuous Wilson line.

As examples we studied a \mathbb{Z}_6 -II and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ heterotic orbifold. For the \mathbb{Z}_6 -II prescriptions in rotational as well as shift embedding are given. The connection between the two approaches is further illustrated. Although the electroweak symmetry breaking can be achieved by a continuous Wilson line this model has some shortcomings. There are additional non Abelian symmetries which are also broken only by the continuous Wilson line. The situation improves for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model. Apart from the Standard Model group all non Abelian groups are broken at the compactification scale by orbifold and discrete Wilson lines. A continuous Wilson line providing exactly electroweak symmetry breaking has been identified.

Many questions are left open for future research. In our models for electroweak symmetry breaking we focused on the gauge sector only. A realistic model should provide also the

matter spectrum with the correct quark and lepton masses. The geometric understanding of spontaneous symmetry breaking, discussed in our paper, can serve the intuition in the search for realistic models.

Acknowledgements

This work was partially supported by the European community's 6th framework programs MRTN-CT-2004-503369 "Quest for Unification" and MRTN-CT-2004-005104 "Forces Universe". We would like to thank Patrick Vaudrevange for useful discussions.

A Rotational versus Shift Embedding in the \mathbb{Z}_3 Case

In the following, we will discuss how our considerations in section 2 are modified if we study a \mathbb{Z}_3 orbifold instead of \mathbb{Z}_2 . For simplicity, we consider just two extra dimension and take for the bulk group $SU(3)$. The extra dimensions are compactified on an $SU(3)$ root lattice. First we will embed the \mathbb{Z}_3 as a rotation into $SU(3)$ and discuss the continuous Wilson lines. This will be akin to the discussion of [44] where the same embedding is given for $SU(3)$ subgroups of $E_8 \times E_8$. Afterwards we will consider these continuous Wilson lines in shift embeddings.

A.1 Rotational Embedding

The discussion here will be on a schematic level and we will not construct the full algebra automorphism corresponding to the embedding of \mathbb{Z}_3 . The six roots of $SU(3)$ are

$$\alpha_1 = (\sqrt{2}, 0) \quad , \quad \alpha_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right) \quad , \quad -\alpha_1 \quad , \quad -\alpha_2 \quad , \quad \pm(\alpha_1 + \alpha_2). \quad (105)$$

We specify the rotational embedding of \mathbb{Z}_3 by the rotation matrix

$$\Theta = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad (106)$$

acting on the vector of Cartan generators $H = (H_1, H_2)^T$. The embedding appears as a cyclic permutation within two subsets of the roots consisting of three roots each

$$\alpha_1 \xrightarrow{\Theta} \alpha_2 \xrightarrow{\Theta} -\alpha_1 - \alpha_2 \quad , \quad -\alpha_1 \xrightarrow{\Theta} -\alpha_2 \xrightarrow{\Theta} \alpha_1 + \alpha_2. \quad (107)$$

The eigenvalues of Θ are $\exp(\pm 2\pi i/3)$ and hence there are no invariant directions. The Cartan generators H_1 and H_2 are projected out. There are two invariant combinations of root generators (suppressing possible phase factors appearing when lifting the embedding to an algebra automorphism)

$$E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1 - \alpha_2} \quad , \quad E_{-\alpha_1} + E_{-\alpha_2} + E_{\alpha_1 + \alpha_2}. \quad (108)$$

From these we can construct two commuting hermitian operators which serve as new Cartan generators

$$H_1^{new} = E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1-\alpha_2} + E_{-\alpha_1} + E_{-\alpha_2} + E_{\alpha_1+\alpha_2}, \quad (109)$$

$$H_2^{new} = i(E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1-\alpha_2} - E_{-\alpha_1} - E_{-\alpha_2} - E_{\alpha_1+\alpha_2}). \quad (110)$$

The unbroken gauge group is

$$U(1)^2.$$

As explained in [44] the continuous Wilson line can be parameterised along $SU(3)$ weight directions

$$W = \lambda_1 w_1 \cdot H + \lambda_2 w_2 \cdot H, \quad (111)$$

where w_i are the fundamental weights which are dual to the lattice vectors α_i

$$w_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right), \quad w_2 = \left(0, \frac{2}{\sqrt{6}} \right). \quad (112)$$

The two cycles of the compactification torus are identified by the \mathbb{Z}_3 orbifold action. Therefore on the other cycle the Wilson line has to be the \mathbb{Z}_3 image

$$\Theta W = \lambda_1 (w_2 - w_1) \cdot H - \lambda_2 w_1 \cdot H, \quad (113)$$

Already if only one of the λ_i is neither zero nor an integer $SU(3)$ is completely broken in the presence of a continuous Wilson line.

Before reproducing this result in shift embedding let us also consider the geometric picture for the symmetry breaking due to continuous Wilson lines. To this end, we first specify the fixed points within the compact space. In order to avoid confusion with the gauge group we choose non normalised basis vectors for the compactification space

$$e_1 = R\alpha_1, \quad e_2 = R\alpha_2, \quad (114)$$

where α_i are the $SU(3)$ root vectors introduced earlier. The \mathbb{Z}_3 action on the compactification lattice is given by the same Θ as in (106). There are three \mathbb{Z}_3 fixed points on the torus

$$P_1 = O, \quad P_2 = \frac{2R}{3}\alpha_1 + \frac{R}{3}\alpha_2, \quad P_3 = \frac{R}{3}\alpha_1 + \frac{2R}{3}\alpha_2. \quad (115)$$

The point P_1 is a fixed point in the complex plane, whereas the \mathbb{Z}_3 image of P_2 (P_3) has to be shifted by $R\alpha_1$ ($R\alpha_1 + R\alpha_2$) in order to obtain the preimage. This means that the projections are with respect to Θ at P_1 w.r.t. Θ times the adjoint action of $\exp[2\pi i W]$ at P_2 and w.r.t. Θ times the adjoint action of $\exp[-2\pi i \Theta^2 W]$ at P_3 . For simplicity let us consider the case $\lambda_2 = 0$. The algebra elements invariant under the local projections are

$$(\Theta^2 w_1 = -w_2)$$

$$\begin{aligned}
& E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1-\alpha_2} + \text{h.c.}, \\
& i(E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1-\alpha_2} - \text{h.c.}) \quad \text{at } P_1, \\
& E_{\alpha_1} + E_{\alpha_2} + e^{-2\pi i \lambda_1} E_{-\alpha_1-\alpha_2} + \text{h.c.}, \\
& i(E_{\alpha_1} + E_{\alpha_2} + e^{-2\pi i \lambda_1} E_{-\alpha_1-\alpha_2} - \text{h.c.}) \quad \text{at } P_2, \\
& E_{\alpha_1} + e^{2\pi i \lambda_1} E_{\alpha_2} + E_{-\alpha_1-\alpha_2} + \text{h.c.}, \\
& i(E_{\alpha_1} + e^{2\pi i \lambda_1} E_{\alpha_2} + E_{-\alpha_1-\alpha_2} - \text{h.c.}) \quad \text{at } P_3.
\end{aligned} \tag{116}$$

At each of the fixed points there is a $U(1)^2$ gauge symmetry. For generic λ_1 these $U(1)$ s are all embedded differently into the bulk $SU(3)$ and the gauge group is broken completely.

A.2 Shift Embedding

As in the \mathbb{Z}_2 case we use the space group notation. The orbifold is taken to be shift embedded

$$(\theta, 0) : \begin{aligned} H_i & \rightarrow H_i, & i = 1, \dots, r \\ E_{\alpha_k} & \rightarrow \exp\{2\pi i \alpha_k \cdot V\} E_{\alpha_k}, & k = 1, \dots, \dim(G) - r \end{aligned} \tag{117}$$

i.e. we embed the orbifold as the adjoint action with

$$e^{2\pi i V^i H_i}.$$

For the projections at the other two fixed points we also need

$$(\theta, R\alpha_1) : \begin{aligned} H_i & \rightarrow t H_i t^{-1}, & i = 1, \dots, r \\ E_{\alpha_k} & \rightarrow \exp\{2\pi i \alpha_k \cdot V\} t E_{\alpha_k} t^{-1}, & k = 1, \dots, \dim(G) - r \end{aligned} \tag{118}$$

and

$$(\theta, R\alpha_1 + R\alpha_2) : \begin{aligned} H_i & \rightarrow t_1 H_i t_1^{-1}, & i = 1, \dots, r \\ E_{\alpha_k} & \rightarrow \exp\{2\pi i \alpha_k \cdot V\} t_1 E_{\alpha_k} t_1^{-1}, & k = 1, \dots, \dim(G) - r \end{aligned} \tag{119}$$

Again, we describe a Wilson line by some general adjoint transformation with constant group elements t, t_1 . From the orbifold identification $-R(\alpha_1 + \alpha_2) = \theta^2 R\alpha_1$ one obtains the condition

$$t_1 = e^{4\pi i V^i H_i} t^{-1} e^{-4\pi i V^i H_i}. \tag{120}$$

The condition that the space group element leaving the point P_2 invariant is a \mathbb{Z}_3 action yields the consistency condition

$$t \left(e^{2\pi i V^i H_i} t e^{-2\pi i V^i H_i} \right) \left(e^{4\pi i V^i H_i} t e^{-4\pi i V^i H_i} \right) = 1, \tag{121}$$

which should also hold with arbitrary permutations of the three factors (\mathbb{Z}_3 is Abelian). Then, the relation (120) implies that the embedding of the element leaving P_3 fixed is

consistent. In addition, t should be unitary in order to ensure that the orbifold maps hermitian operators to hermitian operators.

Now, let us discuss solutions to (121). If t is an element of the Cartan subgroup the condition implies $t^3 = 1$. This corresponds to a discrete Wilson line and will not be discussed further here. For the solution corresponding to a continuous Wilson line we make the ansatz

$$t = e^{2\pi i(T+T^\dagger)} \quad (122)$$

ensuring unitarity of t . Further we choose T to be an eigenstate of the orbifold action, whose eigenvalue we take without loss of generality to be given by

$$e^{2\pi i V^i H_i} T e^{-2\pi i V^i H_i} = e^{\frac{2\pi i}{3}} T. \quad (123)$$

In difference to the \mathbb{Z}_2 case the eigenvalue is not real and T^\dagger transforms with a different phase. Therefore, the \mathbb{Z}_3 case differs from the \mathbb{Z}_2 case since now we also need to impose

$$[T, T^\dagger] = 0. \quad (124)$$

In particular it will be impossible to equate T with a single root operator, now. This will be important when we explicitly connect this construction to the earlier discussed rotational embedding. If all the above conditions are satisfied (121) is solved thanks to the identity

$$1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}} = 0.$$

As in the \mathbb{Z}_2 case we can write the action of the space group element in (118) as a shift embedding w.r.t. to a conjugated Cartan Weyl basis

$$\hat{H}_i = t^{2/3} \left(e^{2\pi i V^i H_i} t e^{-2\pi i V^i H_i} \right)^{1/3} H_i \left(e^{2\pi i V^i H_i} t e^{-2\pi i V^i H_i} \right)^{-1/3} t^{-2/3}, \quad (125)$$

$$\hat{E}_\alpha = t^{2/3} \left(e^{2\pi i V^i H_i} t e^{-2\pi i V^i H_i} \right)^{1/3} E_\alpha \left(e^{2\pi i V^i H_i} t e^{-2\pi i V^i H_i} \right)^{-1/3} t^{-2/3}. \quad (126)$$

Similar considerations apply to the fixed point P_3 . This reproduces the geometrical picture that the unbroken gauge group at all fixed points is the same, but the embedding into the bulk group differs from fixed point to fixed point.

Let us now come back to our particular $SU(3)$ example from the previous section. We choose the shift vector

$$V = \frac{1}{3}\alpha_1. \quad (127)$$

This implies that under the orbifold the Cartan generators are invariant, E_{α_1} , E_{α_2} and $E_{-\alpha_1-\alpha_2}$ transform with a phase $\exp\{-2\pi i/3\}$ and the remaining root generators transform with the complex conjugated phase. Thus, none of the root generators is invariant under the orbifold and the unbroken gauge group is

$$U(1)^2. \quad (128)$$

Now, let us turn on a continuous Wilson line. As in the \mathbb{Z}_2 case we seek a maximal set of hermitian mutually commuting generators which transform non trivially under the orbifold. In addition, terms within those generators transforming differently under the orbifold have to commute. In our case, such a set is given by

$$C_1 = E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1-\alpha_2} + E_{-\alpha_1} + E_{-\alpha_2} + E_{\alpha_1+\alpha_2}, \quad (129)$$

$$C_2 = i(E_{\alpha_1} + E_{\alpha_2} + E_{-\alpha_1-\alpha_2} - E_{-\alpha_1} - E_{-\alpha_2} - E_{\alpha_1+\alpha_2}), \quad (130)$$

and the continuous Wilson line can be parameterised e.g. as

$$T + T^\dagger = \lambda_1 C_1 + \lambda_2 C_2. \quad (131)$$

Neither C_1 nor C_2 commute with any combination of the Cartan generators and hence the gauge symmetry is broken completely as long as at least one of the λ_i is non vanishing.

Let us discuss the connection to rotational embeddings, now. Due to their defining properties C_1 and C_2 can serve as an alternative set of Cartan generators. Under the originally shift embedded orbifold they transform as

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \longrightarrow \Theta \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (132)$$

where Θ is the same rotation matrix as in (106). Thus w.r.t. that set of Cartan generators the shift embedded orbifold looks the same as the rotationally embedded orbifold considered in the previous section and the description of continuous Wilson lines agrees with the previous one.

The presentation of \mathbb{Z}_2 and \mathbb{Z}_3 should suffice to render the discussion of continuous Wilson lines in any \mathbb{Z}_N orbifold straightforward.

B Invariant Combinations of Root Generators

In this appendix we derive equation (34). By using formulæ as given in appendix A of [45] one can compute the transformed E_α and E_β

$$tE_\alpha t^{-1} = \cos(\lambda_2 N_{\gamma,\alpha}) \cos(\lambda_1 N_{\delta,\alpha}) E_\alpha + i \sin(\lambda_2 N_{\gamma,\alpha}) \cos(\lambda_1 N_{\delta,\alpha}) E_{\gamma+\alpha} \\ + i \cos(\lambda_2 N_{\gamma,\alpha+\delta}) \sin(\lambda_1 N_{\delta,\alpha}) E_{\alpha+\delta} - \sin(\lambda_2 N_{\gamma,\alpha+\delta}) \sin(\lambda_1 N_{\delta,\gamma}) E_\beta, \quad (133)$$

$$tE_\beta t^{-1} = \cos(\lambda_2 N_{-\gamma,\beta}) \cos(\lambda_1 N_{-\delta,\beta}) E_\beta + i \sin(\lambda_2 N_{-\gamma,\beta}) \cos(\lambda_1 N_{-\delta,\beta}) E_{\alpha+\delta} \\ + i \cos(\lambda_2 N_{-\gamma,\beta-\delta}) \sin(\lambda_1 N_{-\delta,\beta}) E_{\alpha+\gamma} - \sin(\lambda_2 N_{-\gamma,\beta-\delta}) \sin(\lambda_1 N_{-\delta,\beta}) E_\alpha, \quad (134)$$

where the N_{ρ_1,ρ_2} are the E_6 structure constants

$$[E_{\rho_1}, E_{\rho_2}] = N_{\rho_1,\rho_2} E_{\rho_1+\rho_2}, \quad (135)$$

and we have used the facts that N_{ρ_1,ρ_2} is non vanishing only if $\rho_1 + \rho_2$ is a root and

$$\alpha + \gamma + \delta - \beta = 0. \quad (136)$$

There are several consistency conditions which the structure constants have to satisfy. These can be found in [54]. We list them as they appear for simply laced groups in which case the non vanishing structure constants can be chosen as ± 1 (Φ denotes the set of all root vectors):

- (i) $N_{\rho,\sigma} = -N_{\sigma,\rho}$ for $\rho, \sigma \in \Phi$
- (ii) $N_{\rho_1,\rho_2} = N_{\rho_2,\rho_3} = N_{\rho_3,\rho_1}$ if $\rho_1, \rho_2, \rho_3 \in \Phi$ and $\rho_1 + \rho_2 + \rho_3 = 0$.
- (iii) $N_{\rho,\sigma} = -N_{-\rho,-\sigma}$ for $\rho, \sigma \in \Phi$.
- (iv) $N_{\rho_1,\rho_2}N_{\rho_3,\rho_4} + N_{\rho_2,\rho_3}N_{\rho_1,\rho_4} + N_{\rho_3,\rho_1}N_{\rho_2,\rho_4} = 0$ if $\rho_1, \rho_2, \rho_3, \rho_4 \in \Phi$ and $\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ and no pairs are opposite.

These conditions follow essentially from the Jacobi identity (see [54]). We can use them to find a consistent choice for the constants appearing in (133) and (134). By taking (ρ_1, ρ_2, ρ_3) as $(\alpha, \gamma, \delta - \beta)$ and $(\alpha + \delta, \gamma, -\beta)$ one finds from (ii)

$$N_{\gamma,\delta-\beta} = N_{\alpha,\gamma} \text{ and } N_{\alpha+\delta,\gamma} = N_{\gamma,-\beta}. \quad (137)$$

Eq. (136) applied to (iv) yields

$$N_{\alpha,\gamma}N_{\delta,-\beta} + N_{\delta,\alpha}N_{\gamma,-\beta} = 0, \quad (138)$$

which we choose to solve by (using also (iii))

$$N_{\alpha,\gamma} = N_{\delta,-\beta} = N_{\delta,\alpha} = N_{-\gamma,\beta} = 1. \quad (139)$$

After all the structure constants occurring in (133) and (134) have been fixed we find

$$\begin{aligned} t(E_\alpha - E_\beta)t^{-1} &= (\cos \lambda_1 \cos \lambda_2 + \sin \lambda_1 \sin \lambda_2)(E_\alpha - E_\beta) \\ &\quad + i(\sin \lambda_1 \cos \lambda_2 - \sin \lambda_2 \cos \lambda_1)(E_{\alpha+\delta} - E_{\alpha+\gamma}), \end{aligned} \quad (140)$$

establishing that $E_\alpha - E_\beta$ is indeed invariant if $\lambda_1 = \lambda_2$. Note also, that the present calculation shows $E_\alpha + E_\beta$ to be invariant if $\lambda_1 = -\lambda_2$.

References

- [1] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B **261** (1985) 678.
- [2] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B **274** (1986) 285.
- [3] L. E. Ibáñez, H. P. Nilles and F. Quevedo, Phys. Lett. B **187** (1987) 25.
- [4] L. E. Ibáñez, J. E. Kim, H. P. Nilles and F. Quevedo, Phys. Lett. B **191**, 282 (1987).
- [5] D. Bailin, A. Love and S. Thomas, Phys. Lett. B **188** (1987) 193.

- [6] D. Bailin, A. Love and S. Thomas, Nucl. Phys. B **288** (1987) 431.
- [7] D. Bailin, A. Love and S. Thomas, Phys. Lett. B **194** (1987) 385.
- [8] D. Bailin, A. Love and S. Thomas, Mod. Phys. Lett. A **3** (1988) 167.
- [9] L. E. Ibáñez, J. Mas, H. P. Nilles and F. Quevedo, Nucl. Phys. B **301** (1988) 157.
- [10] J. A. Casas, E. K. Katehou and C. Muñoz, Nucl. Phys. B **317** (1989) 171.
- [11] J. A. Casas and C. Muñoz, Phys. Lett. B **209** (1988) 214.
- [12] J. A. Casas and C. Muñoz, Phys. Lett. B **214** (1988) 63.
- [13] Y. Katsuki, Y. Kawamura, T. Kobayashi and N. Ohtsubo, Phys. Lett. B **212** (1988) 339.
- [14] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo and K. Tanioka, Prog. Theor. Phys. **82** (1989) 171.
- [15] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono and K. Tanioka, Phys. Lett. B **227** (1989) 381.
- [16] K. w. Hwang and J. E. Kim, Phys. Lett. B **540** (2002) 289 [arXiv:hep-ph/0205093].
- [17] J. E. Kim, Phys. Lett. B **564** (2003) 35 [arXiv:hep-th/0301177].
- [18] K. S. Choi, K. Hwang and J. E. Kim, Nucl. Phys. B **662** (2003) 476 [arXiv:hep-th/0304243].
- [19] K. S. Choi and J. E. Kim, Phys. Lett. B **567** (2003) 87 [arXiv:hep-ph/0305002].
- [20] J. E. Kim, JHEP **0308** (2003) 010 [arXiv:hep-ph/0308064].
- [21] J. E. Kim, Phys. Lett. B **591** (2004) 119 [arXiv:hep-ph/0403196].
- [22] J. Giedt, Annals Phys. **297** (2002) 67 [arXiv:hep-th/0108244].
- [23] J. Giedt, Nucl. Phys. B **671** (2003) 133 [arXiv:hep-th/0301232].
- [24] J. Giedt, G. L. Kane, P. Langacker and B. D. Nelson, Phys. Rev. D **71** (2005) 115013 [arXiv:hep-th/0502032].
- [25] K. S. Choi, S. Groot Nibbelink and M. Trapletti, JHEP **0412** (2004) 063 [arXiv:hep-th/0410232].
- [26] F. Quevedo, arXiv:hep-th/9603074.
- [27] A. E. Faraggi, Nucl. Phys. B **387** (1992) 239 [arXiv:hep-th/9208024].

- [28] A. E. Faraggi, Phys. Lett. B **278** (1992) 131.
- [29] A. E. Faraggi, C. Kounnas, S. E. M. Nooij and J. Rizos, Nucl. Phys. B **695** (2004) 41 [arXiv:hep-th/0403058].
- [30] R. Donagi and A. E. Faraggi, Nucl. Phys. B **694** (2004) 187 [arXiv:hep-th/0403272].
- [31] A. E. Faraggi, arXiv:hep-th/0411118.
- [32] Y. Kawamura, Prog. Theor. Phys. **105** (2001) 999 [arXiv:hep-ph/0012125].
- [33] Y. Kawamura, Prog. Theor. Phys. **105** (2001) 691 [arXiv:hep-ph/0012352].
- [34] G. Altarelli and F. Feruglio, Phys. Lett. B **511** (2001) 257 [arXiv:hep-ph/0102301].
- [35] T. Kawamoto and Y. Kawamura, arXiv:hep-ph/0106163.
- [36] A. Hebecker and J. March-Russell, Nucl. Phys. B **613** (2001) 3 [arXiv:hep-ph/0106166].
- [37] T. Asaka, W. Buchmüller and L. Covi, Phys. Lett. B **523** (2001) 199 [arXiv:hep-ph/0108021].
- [38] L. J. Hall and Y. Nomura, Annals Phys. **306** (2003) 132 [arXiv:hep-ph/0212134].
- [39] S. Förste, H. P. Nilles, P. K. S. Vaudrevange and A. Wingerter, Phys. Rev. D **70** (2004) 106008 [arXiv:hep-th/0406208].
- [40] T. Kobayashi, S. Raby and R. J. Zhang, Phys. Lett. B **593** (2004) 262 [arXiv:hep-ph/0403065].
- [41] T. Kobayashi, S. Raby and R. J. Zhang, Nucl. Phys. B **704** (2005) 3 [arXiv:hep-ph/0409098].
- [42] W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, Nucl. Phys. B **712** (2005) 139 [arXiv:hep-ph/0412318].
- [43] W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, arXiv:hep-ph/0511035.
- [44] L. E. Ibáñez, H. P. Nilles and F. Quevedo, Phys. Lett. B **192** (1987) 332.
- [45] S. Förste, H. P. Nilles and A. Wingerter, Phys. Rev. D **72** (2005) 026001 [arXiv:hep-th/0504117].
- [46] A. Font, L. E. Ibáñez, H. P. Nilles and F. Quevedo, Nucl. Phys. B **307** (1988) 109 [Erratum-ibid. B **310** (1988) 764].
- [47] A. Font, L. E. Ibáñez, H. P. Nilles and F. Quevedo, Phys. Lett. **210B** (1988) 101 [Erratum-ibid. B **213** (1988) 564].

- [48] F. Gmeiner, S. Groot Nibbelink, H. P. Nilles, M. Olechowski and M. G. A. Walter, Nucl. Phys. B **648** (2003) 35 [arXiv:hep-th/0208146].
- [49] A. Hebecker and M. Ratz, Nucl. Phys. B **670** (2003) 3 [arXiv:hep-ph/0306049].
- [50] A. Wingerter, PhD Thesis, Bonn University, June 2005, http://hss.ulb.uni-bonn.de:90/ulb_bonn/diss_online/math_nat_fak/2005/wingerter_akin/index.htm
- [51] H. P. Nilles, arXiv:hep-th/0410160.
- [52] A. N. Schellekens and N. P. Warner, Nucl. Phys. B **308** (1988) 397.
- [53] P. Bouwknecht, J. Math. Phys. **30** (1989) 571.
- [54] R. W. Carter, “*Simple Groups of Lie Type*,” John Wiley & Sons (1972) 331 p. (Pure and Applied Mathematics 28).
- [55] A. Zee, “*Quantum field theory in a nutshell*,” Princeton Univ. Pr. (2003) 518 p.